TWO q–IDENTITIES FROM THE THEORY OF FOUNTAINS AND HISTOGRAMS PROVED WITH A TRI–DIAGONAL DETERMINANT

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Abstract

Two identities required in the theory of fountains and histograms are easily proved by expanding a tri–diagonal determinant (reminiscent of Schur's) in two different ways.

We consider the following infinite tri-diagonal determinant (elements not displayed are zero)

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & & \overbrace{0\dots0}^{p-2} & & & xq^1 & & & & \dots \\ -1 & 1 & & & 0\dots0 & & xq^2 & & & \dots \\ & -1 & 1 & & & 0\dots0 & & xq^3 & & \dots \\ & & -1 & 1 & & & 0\dots0 & & xq^4 & \dots \\ & & & \ddots & \ddots & & \ddots & & & \end{vmatrix}.$$

Schur, when providing his proof of the Rogers–Ramanujan identities in 1917 [3] used a similar determinant; since I am advocating that Schur's work deserves to be better known, I use the name $\mathsf{Schur}(x)$. This short note shows that two identities that were required in the study of fountains and histograms [1] are most easily proved by expanding the determinant in two different ways.

Expanding the determinant with respect to the first column ("top–recursion") we get

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + (-1)^p xq\ \mathsf{Schur}(xq^p).$$

Setting

$$\mathsf{Schur}(x) = \sum_{n\ge0} a_n x^n,$$

we get, upon comparing coefficients,

$$a_n = q^n a_n + (-1)^p q^{1+p(n-1)} a_{n-1} = \frac{(-1)^p q^{1+p(n-1)}}{1-q^n}.$$

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Since $a_0 = 1$, iteration leads to

$$a_n = \frac{q^{n+p\binom{n}{2}}(-1)^{pn}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Therefore

$$\begin{aligned}\mathsf{Schur}((-q)^{p-1}) &= \sum_{n\ge 0} \frac{(-1)^n q^{n+p\binom{n}{2}}}{(1-q)(1-q^2)\dots(1-q^n)} q^{(p-1)n} \\ &= \sum_{n\ge 0} \frac{(-1)^n q^{p\binom{n+1}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.\end{aligned}$$

Now consider the *finite* determinants $\mathsf{Schur}_n(x)$, obtained from $\mathsf{Schur}(x)$ by taking the first n rows and columns. Expanding this determinant with respect to the last row ("bottom–recursion") we get

$$\mathsf{Schur}_n(x) = \mathsf{Schur}_{n-1}(x) + (-1)^p x q^{n-p+1} \,\mathsf{Schur}_{n-p}(x).$$

In particular,

$$\mathsf{Schur}_n((-q)^{p-1}) = \mathsf{Schur}_{n-1}((-q)^{p-1}) - q^n \,\mathsf{Schur}_{n-p}((-q)^{p-1}),$$

and $\mathsf{Schur}_j((-q)^{p-1}) = 1$ for $j = 0, \dots, p-1$. The quantities $\mathsf{Schur}_n((-q)^{p-1})$ were called E_n in [1] (with matching initial conditions $E_j = 1$ for $j = 0, \dots, p-1$). Whence we proved

$$\lim_{m\to\infty} E_m = \sum_{n\ge 0} \frac{(-1)^n q^{p\binom{n+1}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Merlini and Sprugnoli had asked for a direct proof, which was given in [2], by showing an explicit form for E_m. The present proof avoids this and is thus simpler.

A second (similar) formula was also requested, namely

$$\lim_{m\to\infty} D_m = \sum_{n\ge 0} \frac{(-1)^n q^{n+p\binom{n}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

for $D_n = D_{n-1} - q^n D_{n-p}$ and (different) initial values $D_j = 1 - \sum_{i=1}^{j} q^i$ for $j = 0, \dots, p-1$. This follows immediately by setting $D_n = \mathsf{Schur}_{n+p-1}((-1)^{p-1})$.

References

[1] D. Merlini and R. Sprugnoli, *Fountains and histograms*, J. Algorithms **44** (2002), no. 1, 159–176.

[2] P. Paule and H. Prodinger, *Fountains, histograms, and q–identities*, Discrete Mathematics and Theoretical Computer Science **6** (2003), 101–106.

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