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NOTE

A SHORT PROOF FOR A PARTITION IDENTITY OF HWANG AND WEI

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In Volume 46 of this journal [1] Hwang and Wei prove the following identity:

$$\sum_{p\in P}\prod_{i=1}^{m}\binom{n_i+1-k_i}{k_i}=\sum_{j\geq 0}\binom{j+m-2}{m-2}\binom{n+1-k-2j}{k-2j}$$

for integers n_i with $n_1+n_2+\cdots+n_m=n$ $(m\geq 2)$, where $p=(k_1,\ldots,k_m)$ runs through the set P of all partitions $k=k_1+k_2+\cdots+k_m$ of the nonnegative integer k into m non-negative integers k_i. We give here a very short proof for this identity, using generating functions.

We set

$$A_n(z)=\sum_{k\geq 0}\binom{n+1-k}{k}z^k.$$

Then it follows from Riordan [2, p. 154, 2c; note the typos!], that

$$A_n(z)=\mu^{-1}\cdot\left(\frac{-2z}{1-\mu}\right)^{n+2}=\mu^{-1}\cdot\left(\frac{1+\mu}{2}\right)^{n+2}\quad\text{with } \mu=(1+4z)^{1/2}. \tag{1}$$

The identity in question is equivalent to

$$\begin{aligned}\prod_{i=1}^{m}A_{n_i}(z)&=\sum_{j\geq 0}\binom{j+m-2}{m-2}z^{2j}A_{n-4j}(z)\\&=\sum_{j\geq 0}\binom{-m+1}{j}(-z^2)^jA_{n-4j}(z).\end{aligned}$$

With $n=n_1+n_2+\cdots+n_m$ and formula (1), we have to show

$$\mu^{-m}\left(\frac{1+\mu}{2}\right)^{n+2m}=\mu^{-1}\left(\frac{1+\mu}{2}\right)^{n+2}\left(1-\frac{(1-\mu)^4}{16z^2}\right)^{-m+1}.$$

Regarding $4z = -(1+\mu)(1-\mu)$ it follows that

$$1-\frac{(1-\mu)^4}{16z^2}=1-\frac{(1-\mu)^2}{(1+\mu)^2}=\frac{4\mu}{(1+\mu)^2}$$

from which the desired identity is immediate.

References

[1] F.K. Hwang and V.K. Wei, A partition identity, Discrete Math. 46 (1983) 323–326.
[2] J. Riordan, Combinatorial Identities (Wiley, New York, 1968).