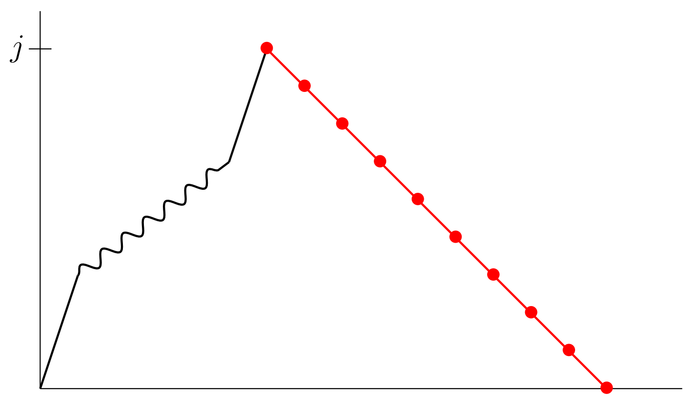
ON THE ENUMERATION OF HOPPY'S WALKS

HELMUT PRODINGER

Deng and Mansour [1] introduce a rabbit named Hoppy and let him move according to certain rules. At that stage, we don't need to know the rules. Eventually, the enumeration problem is one about k-Dyck paths. The up-steps are $(1, k)$ and the down-steps are $(1, -1)$.



The question is about the length of the sequence of down-steps printed in red. Or, phrased differently, how many k-Dyck paths end on level j, after m up-steps, the last step being an up-step. The recent paper [4] contains similar computations, although without the restriction that the last step must be an up-step.

Counting the number of up-steps is enough, since in total, there are $m + km = (k + 1)m$ steps. The original description of Deng and Mansour is a reflection of this picture, with up-steps of size 1 and down-steps of sice $-k$, but we prefer it as given here, since we are going to use the adding-a-new-slice method, see [2, 3]. A slice is here a run of down-steps, followed by an up-step. The first up-step is treated separately, and then $m - 1$ new slices are added. We keep track of the level after each slice, using a variable u. The variable z is used to count the number of up-steps.

Deng and Mansour work out a formula which comprises $O(m)$ terms. Our method leads only to a sum of $O(j)$ terms.

The following substitution is essential for adding a new slice:

$$u^j \longrightarrow z\sum_{0\le h\le j} u^{h+k} = \frac{zu^k}{1-u}(1-u^{j+1}).$$

Now let $F_m(z,u)$ we the generating function according to m runs of down-steps. The substitution leads to

$$F_{m+1}(z,u) = \frac{zu^k}{1-u}F_m(z,1) - \frac{zu^{k+1}}{1-u}F_m(z,u), \quad F_0(z,u) = zu^k.$$

Let $F = \sum_{m\ge 0} F_m$, then

$$F(z,u) = zu^k + \frac{zu^k}{1-u}F(z,1) - \frac{zu^{k+1}}{1-u}F(z,u),$$

or

$$F(z,u)\frac{1-u+zu^{k+1}}{1-u} = zu^k + \frac{zu^k}{1-u}F(z,1).$$

The equation $1-u+zu^{k+1}=0$ is famous when enumerating $(k+1)$-ary trees. Its relevant combinatorial solution (also the only one being analytic at the origin) is

$$\overline{u} = \sum_{\ell\ge 0}\frac{1}{1+\ell(k+1)}\binom{1+\ell(k+1)}{\ell}z^\ell.$$

Since $u-\overline{u}$ is a factor of the LHS, is must also be a factor of the RHS, and we can compute (by dividing out the factor $(u-\overline{u})$) that

$$\frac{zu^k(1-u+F(z,1))}{u-\overline{u}} = -zu^k.$$

Thus

$$F(z,u) = zu^k\frac{\overline{u}-u}{1-u+zu^{k+1}}.$$

The first factor has even a combinatorial interpretation, as a description of the first step of the path. It is also clear from this that the level reached is $\ge k$ after each slice. We don't care about the factor zu^k anymore, as it produces only a simple shift. The main interest is now how to get to the coefficients of

$$\frac{\overline{u}-u}{1-u+zu^{k+1}}$$

in an efficient way. There is also the formula

$$1-u+zu^{k+1} = (\overline{u}-u)\Big(1-z\frac{u^{k+1}-\overline{u}^{k+1}}{u-\overline{u}}\Big),$$

but it does not seem to be useful here.

First we deal with the denominators

$$S_j := [u^j]\frac{1}{1-u+zu^{k+1}} = \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}z^m.$$

One way to see this formula is to prove by induction that the sums S_j satisfy the recursion

$$S_j - S_{j-1} + zS_{j-k-1} = 0$$

and initial conditions $S_0 = \cdots = S_k = 1$. In [4] such expressions also appear as determinants. Summarizing,

$$\frac{1}{1-u+zu^{k+1}} = \sum_{m\ge 0}(-1)^m z^m \sum_{j\ge km}\binom{j-km}{m}u^j.$$

Now we read off coefficients:

$$[u^j]\frac{\overline{u}}{1-u+zu^{k+1}}$$
$$= \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}z^m\sum_{\ell\ge 0}\frac{1}{1+\ell(k+1)}\binom{1+\ell(k+1)}{\ell}z^\ell$$

and further

$$[z^n][u^j]\frac{\overline{u}}{1-u+zu^{k+1}}$$
$$= \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}\frac{1}{1+(n-m)(k+1)}\binom{1+(n-m)(k+1)}{n-m}.$$

The final answer to the Deng-Mansour enumeration (without the shift) is

$$\sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}\frac{1}{1+(n-m)(k+1)}\binom{1+(n-m)(k+1)}{n-m}$$
$$-(-1)^n\binom{j-1-kn}{n}.$$

If one wants to take care of the factor zu^k as well, one needs to do the replacements $n \to n+1$ and $j \to j+k$ in the formula just derived. That enumerates then the k-Dyck paths ending at level j after n up-steps, where the last step is an up-step.

We hope that the methods presented here might be useful for other questions related to ascents/descents.

References

[1] Eva Deng and Toufik Mansour, Three Hoppy path problems and ternary paths, Discrete Applied Mathematics 156, 2008, 770–779.
[2] Flajolet, Philippe and Prodinger, Helmut, Level number sequences for trees, Discrete Mathematics 65, 1987, 149–156.
[3] Helmut Prodinger. Analytic methods. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 173–252. CRC Press, Boca Raton, FL, 2015.
[4] Prodinger, Helmut, On k-Dyck paths with a negative boundary.

Helmut Prodinger, Department of Mathematical Sciences, Stellenbosch University, 7602 Stellenbosch, South Africa
Email address: hproding@sun.ac.za