The Fibonacci Quarterly

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION

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VOLUME 32 NOVEMBER 1994 NUMBER 5

SOME INFORMATION ABOUT THE BINOMIAL TRANSFORM

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(Submitted April 1993)

A few days ago I saw the paper [4]. I think I can make some additional remarks that might not be totally useless for the Fibonacci Community!

Let (a_n) be a given sequence and $s_n = \sum_{k=0}^{n}\binom{n}{k}a_k$. Denoting the respective (ordinary) generating functions by $A(x)$ and $S(x)$, the paper in question mainly deals with the consequences of the formula

$$S(x) = \frac{1}{1-x}A\left(\frac{x}{1-x}\right). \tag{1}$$

Knuth [7] has introduced the *binomial transform* by

$$\hat{a}_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k,$$

and it is clear that this is the situation from above. But Philippe Flajolet and the present writer agreed about ten years ago that there are just *exponential generating functions* hidden! They have a convolution formula

$$c_n = \sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k},$$

and upon choosing the b_k's to be equal to 1, we have the old situation. So, denoting the exponential generating functions by $\overline{A}(x)$ and $\overline{S}(x)$, we have the even simpler formula $\overline{S}(x) = e^x\overline{A}(x)$. This can readily be inverted as $\overline{A}(x) = e^{-x}\overline{S}(x)$, whence

$$a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}s_k.$$

These facts about exponential generating functions are of course folklore; one particular reference is [3].

Flajolet & Richmond [2], Schmid [8], and Kirschenhofer & Prodinger [6] all made heavy use of (1). Schmid observed (among other writers) that an exponential generating function will be transformed into an ordinary generating function by the *Borel transform*.

Now the *generalization*

$$s_n = \sum_{k=0}^{n}\binom{n}{k}b^{n-k}c^k a_k \quad \text{or} \quad S(x) = \frac{1}{1-bx}A\left(\frac{cx}{1-bx}\right)$$

translates into

$$\overline{S}(x) = e^{bx}\overline{A}(cx).$$

Since

$$\overline{A}(x) = e^{-\frac{b}{c}x}\,\overline{S}\left(\frac{x}{c}\right),$$

we find the inversion formula

$$a_n = c^{-n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}b^{n-k}s_k.$$

The discussion in Theorem 2 becomes quite transparent, considering exponential generating functions. It is asked whenever we have

$$F_{pn+r} = \sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^k F_{qk+r},$$

where F_n denote Fibonacci numbers. The exponential generating function of the Fibonacci numbers F_n is

$$\frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}),$$

with the usual $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha = (1-\sqrt{5})/2$. More generally, the sequence F_{pn+r} leads to

$$\frac{1}{\sqrt{5}}\left(\alpha^r e^{\alpha^p x} - \beta^r e^{\beta^p x}\right) = e^{tx}\frac{1}{\sqrt{5}}\left(\alpha^r e^{\alpha^q sx} - \beta^r e^{\beta^q sx}\right),$$

from which we deduce the two equations,

$$\alpha^p = t + \alpha^q s \quad \text{and} \quad \beta^p = t + \beta^q s.$$

Subtracting them, we see that

$$s = \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = \frac{F_p}{F_q}.$$

Further,

$$t = \alpha^p - \alpha^q\frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = (-1)^p\frac{\alpha^{q-p} - \beta^{q-p}}{\alpha^q - \beta^p} = (-1)^p\frac{F_{q-p}}{F_q}.$$

To justify this *equating of coefficients*, we note that the functions $e^{\lambda x}$ are linearly independent; and the other possibility of grouping terms from the left and the right side would lead to the impossible equation $\alpha^r = -\beta^r$.

In [4] there is also the modification: What are the coefficients of

$$T(x) = A\left(\frac{cx}{1-bx}\right)?$$

That means: What is the effect of deleting the first factor? We can answer this much more generally by considering (with an arbitrary complex parameter d),

$$T(x) = \frac{1}{(1-bx)^d}A\left(\frac{cx}{1-bx}\right).$$

In this derivation, we will use the concept of *residues*, interesting *per se*.

We are using the substitution $w = \frac{cx}{1-bx}$ or $x = \frac{w}{c+bw}$. Therefore, $1 - bx = \frac{c}{c+bw}$ and $dx = \frac{c}{(c+bw)^2}dw$; thus,

$$t_n := [x^n]T(x) = \frac{1}{2\pi i}\oint \frac{dx}{x^{n+1}} T(x)$$

$$= \frac{1}{2\pi i}\oint \frac{dx}{x^{n+1}} \frac{(c+bw)^d}{c^d} A(w)$$

$$= \frac{1}{2\pi i}\oint \frac{cdw}{(c+bw)^2} \frac{(c+bw)^{n+1}}{w^{n+1}} \frac{(c+bw)^d}{c^d} A(w)$$

$$= c^{1-d}[w^n](c+bw)^{n+d-1}A(w)$$

$$= \sum_{k=0}^{n} \binom{n+d-1}{n-k} b^{n-k} c^k a_k.$$

Since

$$A(w) = \left(\frac{c}{c+bw}\right)^d T\left(\frac{w}{c+bw}\right),$$

we find in a similar way the inversion formula

$$a_n = c^{-n}\sum_{k=0}^{n} \binom{n+d-1}{n-k} (-1)^{n-k} b^{n-k} t_k.$$

The formula (1) is also useful to deal with Knuth's sum [5, eq. (7.6)]

$$u_n = \sum_{k=0}^{n} \binom{n}{k}\left(-\frac{1}{2}\right)^k \binom{2k}{k}.$$

Since

$$f(x) := \sum_{k\geq 0}\left(-\frac{1}{2}\right)^k \binom{2k}{k} x^k = \sum_{k\geq 0}\left(-\frac{x}{2}\right)^k \binom{2k}{k} = \frac{1}{\sqrt{1+2x}},$$

the generating function of the sequence u_n turns out to be

$$\frac{1}{1-x}\frac{1}{\sqrt{1+2\left(\frac{x}{1-x}\right)}} = \frac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} x^{2n}\binom{2n}{n}4^{-n}.$$

From this, we see that $u_n = 2^{-n}\binom{n}{n/2}$ if n is even, and $u_n = 0$ otherwise.

I communicated this idea to Knuth, and he reported that Herbert Wilf came to this (or a similar) approach independently.

Formula (1) also has a *combinatorial interpretation*. If, for example, $A(x)$ enumerates certain words, so that a_n is the number of words of length n with a certain property, and we perform the operation "fill-in a new letter where and as often as you want," then the new "language" has the generating function $S(x)$. For further details on such *combinatorial constructions*, we refer the reader to [1].

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AMS Classification Number: 05A10

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