A Bijective Proof of an Identity Concerning Nodes of Fixed Degree in Planted Trees

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1 Introduction

We consider *planted plane trees*, which are *rooted*, embedded into the *plane* and enumerated by Catalan numbers $y_n = \frac{1}{n}\binom{2n-2}{n-1}$; see e.g. [2].

In [1], among other things, the following was considered. Denote by y_{nd} the number of (planted plane) trees of size n and root degree d. (It is easy to see that $y_{nd} = \frac{d}{n-1}\binom{2n-3-d}{n-2}$, a *ballot number*.) Also, denote by D_{nd} the total number of nodes of degree d in trees of size n. It was shown by generating function arguments that $D_{nd} = 2\binom{2n-3-d}{n-2}$ whence

$$d\,D_{nd} = 2(n-1)y_{nd}. \qquad (1)$$

It is this formula (1) that we will "explain" combinatorially in the present note.

Example.

Figure 1.

$n = 4,\ y_4 = 5,\ y_{41} = 2,\ y_{42} = 2,\ y_{43} = 1,\ D_{41} = 12,\ D_{42} = 6,\ D_{43} = 2.$

Remark. In the case of labelled trees, the equivalent of formula (1) reads simply $D_{nd} = n\,y_{nd}$.

2 The Bijection

Consider the D_{nd} trees of size n with root node r and a designated node u of degree d.

Let $\mathcal{L}$ denote the set of $d\,D_{nd}$ (partially coloured) trees obtained by assigning (a) the colour red to the node r and to the left-most edge incident with r and (b) the colour blue to the designated node u and to one of the d edges incident with u. (Note that it is possible for a node or an edge to receive both colours.)

Now consider the y_{nd} trees of size n in which the root node has degree d.

Let $\mathcal{R}$ denote the set of $2(n-1)y_{nd}$ (partially coloured) trees obtained by assigning (a) the colour blue to the root node and to the left-most edge incident with the root node and (b) the colour red to one of the $n-1$ edges of the tree and to one of the two nodes incident with this red edge.

To see that the sets $\mathcal{L}$ and $\mathcal{R}$ are equivalent, it suffices to regard each tree in $\mathcal{L}$ as being rooted at its blue node with the blue edge as the left-most edge incident with the root. (Or, conversely, we may regard each tree in $\mathcal{R}$ as being rooted at its red node with the red edge as the left-most edge incident with the root.)

Remark. Let $c_0(=1),\ c_1, c_2, \dots$ denote a sequence of constants and suppose each rooted plane tree T is assigned the weight

$$w(T) = \prod_{v \in V(T)} c_{d(v)}$$

where $V(T)$ denotes the set of nodes of T and $d(v)$ denotes the number of edges incident with node v that lead away from the root of the tree. Let $N_\beta(T)$ denote the number of nodes of degree β in a tree T and let

$$N_n(\alpha, \beta) = \sum N_\beta(T) w(T)$$

where the sum is over all (weighted rooted plane) trees T of size n and root degree α. Then the relation

$$\beta\, c_\beta\, c_{\alpha-1}\, N_n(\alpha,\beta) = \alpha\, c_\alpha\, c_{\beta-1}\, N_n(\beta,\alpha) \qquad (2)$$

follows by a slight extension of the bijective argument used above.

References

[1] A. Meir and J.W. Moon, Survival under random coverings of trees, *Graphs and Combinatorics* 4 (1988), 49–65.

[2] R. Sedgewick and P. Flajolet, *An Introduction to the Analysis of Algorithms*, Addison-Wesley, 1996.