Batcher's odd-even exchange revisited: a generating functions approach

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Abstract

The celebrated odd-even exchange algorithm by Batcher provides the quantity *average number of exchanges*, which was a mystery a few years ago and is still tricky today. We provide an approach that is purely based on generating functions to provide an explicit expression. The asymptotic analysis was done several years ago but never published in a journal and is thus provided here. It is a combination of singularity analysis of generating functions and Mellin transform techniques.

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1. Introduction

Batcher's odd-even merge is a sorting method that is well documented in books and papers, notably in Knuth's monumenal work *The Art of Computer Programming*, Volume 3 [9, 10].

Since it is a bit complicated and long to describe, we refrain from doing this and just mention that its analysis (average number of exchanges, provided that a random permutation is given) boils down to a *lattice path counting problem*: All $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) are considered; for each path, the sum of the vertical weights that it traverses, is recorded. The total sum of these counts, divided by the number of all paths $\binom{2n}{n}$ is denoted by B_n, the average number of exchanges.

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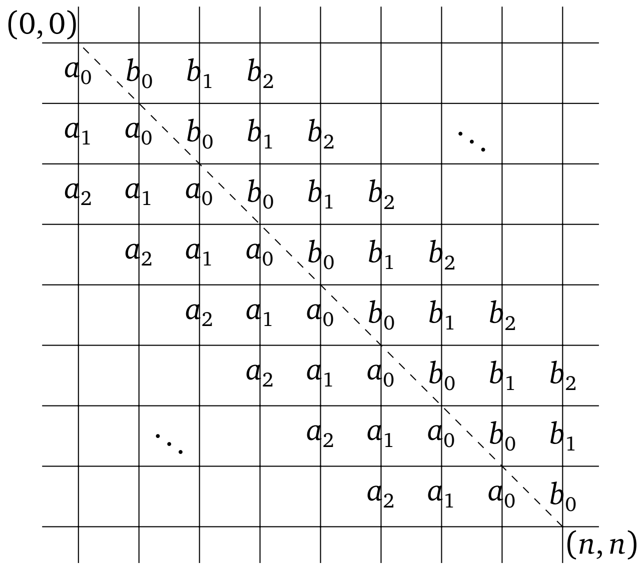

The weights $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ will be discussed in a minute.

The analysis of this quantity was posed as an open problem in [9]; however, Knuth came very close to the solution, as we will soon see. A first complete answer was given by Sedgewick [15], who showed that

$$B_n = \frac{1}{\binom{2n}{n}} \sum_{k \geq 1} \binom{2n}{n-k} (2F(k) + k), \tag{1}$$

where $F(k)$ is the summatory function of $f(j)$, which is the number of ones in the Gray code representation of j:

$$F(k) := \sum_{0 \leq j < k} f(j).$$

More important than the sequence $f(k)$ itself it the sequence of its differences:

$$\theta(k) = f(k) - f(k-1),$$

since these numbers have a number theoretic significance, as discussed later.

The weights in our problem are $a_k = f(k)$ and $b_k = f(k) + 1$.

Sedgewick also provided asymptotics for B_n, by a technique, that was called *Gamma function method* at the period, which is today known as *Mellin transform*; see the survey [2].

Several writers started from the formula (1) and discussed alternative asymptotic methods, see, e. g., [4, 12, 13, 11, 8]. Sedgewick derived this formula by skillful manipulations of binomial coefficients and sums involving them. A more modern approach would be through generating functions, a point of view that is emphasized in the important book [5], see also [6, 14]. Knuth himself considered this problem through generating functions in the first edition [9] and came very close to this formula. However, in the second edition [10], he switched to Sedgewick's approach.

Here, I want to present such a generating function approach, perhaps close to what Knuth had in mind. In a final section, an asymptotic evaluation will be presented, which is probably the best one that exists today; it was already reported in [13], but the EATCS bulletin isn't really a journal, and the approach deserves to be better known.

2. A generating function approach

Instead of summing over all labels of one path, one splits such a path into n copies, where each one carries exactly one of the n labels, and sums these. We call such a path with exactly one vertical label a *decorated path*.

A decorated path goes from $(0,0)$ to (n,n) and carries exactly one (vertical) label. Let $\mathscr{V}$ be the family of all paths $(0,0)$ to (n,n), $\mathscr{D}$ be the family of all paths $(0,0)$ to (n,n), staying on one side of the diagonal, $\mathscr{R}_p$ the family of paths with vertical label a_p, and $\mathscr{S}_p$ the family of paths with vertical label b_p. We write d for a down-step, and h for a horizontal-step.

With roman letters we write the associated ordinary generating functions (only the down-steps, say, are counted).

We have the classical results, with the standard substition $z=\frac{u}{(1+u)^2}$, taken from [1]:

$$W(z)=\frac{1}{\sqrt{1-4z}}=\sum_{n\ge0}\binom{2n}{n}z^n=\frac{1+u}{1-u},$$

$$D(z)=\frac{1-\sqrt{1-4z}}{2z}=\sum_{n\ge0}\frac{1}{n+1}\binom{2n}{n}z^n=1+u.$$

Furthermore, we have, by a decomposition according to returns to the diagonal,

$$\mathscr{R}_p=\mathscr{W}\mathscr{A}_p\mathscr{W},$$

and

$$\mathscr{A}_p=\mathsf{d}\mathscr{D}\mathscr{A}_{p-1}\mathscr{D}\mathsf{h},\quad p\ge1,\quad \mathscr{A}_0=a_p\cdot d\mathscr{D}h.$$

Hence

$$A_p = z(1+u)^2 A_{p-1}, \quad p \geq 1, \quad A_0 = a_p \frac{u}{1+u}.$$

By iteration,

$$A_p = a_p \frac{u^{p+1}}{1+u}$$

and therefore

$$R_p = a_p \frac{u^{p+1}(1+u)}{(1-u)^2};$$

by symmetry

$$S_p = b_p \frac{u^{p+1}(1+u)}{(1-u)^2}.$$

As a check, assume that all weights are equal to one, then

$$\sum_{p\geq 0} (R_p + S_p) = 2\frac{u(1+u)}{(1-u)^3},$$

which checks with the formula

$$\sum_{n\geq 0} n\binom{2n}{n} z^n = \frac{2z}{(1-4z)^{3/2}}.$$

In the Batcher problem, we have $b_p = 1 + a_p$, and $a_p = f(p)$, with $f(k)$ being the number of ones in the Gray code representation of k. Then

$$B(z) := \sum_{p\geq 0} (R_p + S_p) = \frac{u(1+u)}{(1-u)^3} + 2\sum_{p\geq 0} f(p)\frac{u^{p+1}(1+u)}{(1-u)^2}.$$

We know that (explicitly in [9], implicitly in [15])

$$f(p) = [u^p]\frac{1}{1-u}\sum_{k\geq 1} \frac{u^{2^{k-1}}}{1+u^{2^k}}.$$

Note also that

$$\begin{aligned}
\sum_{k\geq 0} \frac{u^{2^k}}{1+u^{2^{k+1}}} &= \sum_{k\geq 0}\sum_{j\geq 0} (-1)^j u^{2^k(2j+1)} \\
&= \sum_{k\geq 0}\sum_{j\geq 0} u^{2^k(4j+1)} - \sum_{k\geq 0}\sum_{j\geq 0} u^{2^k(4j+3)}
\end{aligned}$$

$$= \sum_{n\geq 1} \theta(n) u^n,$$

where

$$\theta(n) = \begin{cases} 1, & \text{for } n = 2^k(4j+1), \\ -1, & \text{for } n = 2^k(4j+3). \end{cases}$$

Therefore

$$\begin{aligned} B(z) &= \frac{u(1+u)}{(1-u)^3} + 2\frac{u(1+u)}{(1-u)^2} \sum_{p\geq 0} u^p \cdot [u^p] \frac{1}{1-u} \sum_{k\geq 1} \frac{u^{2^{k-1}}}{1+u^{2^k}} \\ &= \frac{u(1+u)}{(1-u)^3} + 2\frac{u(1+u)}{(1-u)^3} \sum_{k\geq 0} \frac{u^{2^k}}{1+u^{2^{k+1}}}. \end{aligned}$$

This formula is the one given in [9]; note, however, that a factor $1-u$ seems to be missing in it. Our formula is consistent with Sedgewick's solution, as can be seen as follows:

It is elementary that

$$[z^n] \frac{u(1+u)}{(1-u)^3} = \frac{n}{2} \binom{2n}{n}.$$

This is also

$$\sum_{k\geq} k \binom{2n}{n-k} = \frac{n}{2} \binom{2n}{n}$$

from Sedgewick's formula.

Furthermore,

$$\begin{aligned} [z^n] \frac{u(1+u)}{(1-u)^3} \sum_{k\geq 1} \theta(k) u^k &= \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \frac{u(1+u)}{(1-u)^3} \sum_{k\geq 1} \theta(k) u^k \\ &= \frac{1}{2\pi i} \oint \frac{du(1+u)^{2n}}{u^{n+1}} \frac{u}{(1-u)^2} \sum_{k\geq 1} \theta(k) u^k \\ &= [u^n](1+u)^{2n} \frac{u}{(1-u)^2} \sum_{k\geq 1} \theta(k) u^k \\ &= [u^n](1+u)^{2n} \frac{u}{1-u} \sum_{k\geq 1} f(k) u^k \\ &= [u^n](1+u)^{2n} \sum_{k\geq 1} F(k) u^k \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\ge1} F(k)[u^{n-k}](1+u)^{2n} \\
&= \sum_{k\ge1} F(k)\binom{2n}{n-k}.
\end{aligned}
$$

3. Asymptotics

The general strategy is to use singularity analysis [5, 3]. The goal is to find the local of behavior of

$$\frac{u(1+u)}{(1-u)^3}\sum_{k\ge1}\theta(k)u^k$$

near the dominant singularity $z = \frac{1}{4}$, and then translate it into the behavior of the coefficients. In the u-language, this means the behavior around $u = 1$. To obtain it, the Mellin transform is used. This chain of operations is by now standard. A recent application with a fair amount of technical details provided is [7].

A further substitution $u = e^{-t}$ is necessary for the Mellin transform to work. It means that we have then to investigate the behavior of

$$\frac{e^{-t}(1+e^{-t})}{(1-e^{-t})^3}\sum_{k\ge1}\theta(k)e^{-kt}$$

near $t = 0$. The factor in front is elementary, so we concentrate on the sum

$$V(t) := \sum_{k\ge1}\theta(k)e^{-kt}.$$

By standard rules for the Mellin transform, this is

$$V(s) = \Gamma(s)\sum_{k\ge1}\theta(k)k^{-s}.$$

Note that

$$
\begin{aligned}
\sum_{k\ge1}\theta(k)k^{-s} &= \sum_{m\ge0}\sum_{i\ge0}2^{-ms}(4i+1)^{-s} - \sum_{m\ge0}\sum_{i\ge0}2^{-ms}(4i+3)^{-s} \\
&= \frac{1}{1-2^{-s}}\cdot\frac{1}{4^s}\Big(\zeta(s,\tfrac{1}{4}) - \zeta(s,\tfrac{3}{4})\Big),
\end{aligned}
$$

where the Hurwitz $\zeta(s,a)$-function [16] is defined by

$$\zeta(s,a)=\sum_{n\ge0}(n+a)^{-s} \qquad \text{for } \Re(s)>1.$$

We can write

$$B_n=\frac{n}{2}+\frac{2}{\binom{2n}{n}}C_n,$$

with

$$C_n=\sum_{k\ge1}F(k)\binom{2n}{n-k}=[u^n]\frac{u(1+u)}{(1-u)^3}\sum_{i\ge1}\theta(i)u^i.$$

We easily find that

$$\frac{u(1+u)}{(1-u)^3}\sim\frac{2}{t^3}+\cdots.$$

So we have

$$V^*(s)=\Gamma(s)\cdot\frac{1}{2^s}\cdot\frac{1}{2^s-1}\Big(\zeta\big(s,\tfrac14\big)-\zeta\big(s,\tfrac34\big)\Big).$$

As a consequence (Mellin's inversion formula) we find

$$V(t)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\cdot\frac{1}{2^s}\cdot\frac{1}{2^s-1}\Big(\zeta\big(s,\tfrac14\big)-\zeta\big(s,\tfrac34\big)\Big)t^{-s}dt.$$

The difference of the Hurwitz ζ-functions is an entire function; the integrand has poles at $s=\chi_k:=\frac{2\pi ik}{\log 2}$ and $s=0,-1,-2,\ldots$. We have (see [16])

$$\zeta\big(s,\tfrac14\big)-\zeta\big(s,\tfrac34\big)\sim\frac12+s\Big(2\log\Gamma\big(\tfrac14\big)-\frac12\log2-\log\pi\Big)+\cdots$$

for $s\to0$. Also,

$$\Gamma(s)\sim\frac1s-\gamma+\cdots,$$
$$\frac{1}{2^s-1}\sim\frac{1}{\log2}\cdot\frac1s-\frac12+\cdots,$$
$$(2t)^{-s}\sim1-s\cdot\log(2t)+\cdots,$$

which gives us the residue at $s=0$

$$-\frac{5}{4}-\frac{\gamma}{2\log 2}+2\log_2\Gamma(\tfrac{1}{4})-\log_2\pi-\frac{1}{2}\log_2 t.$$

The residue at $s=\chi_k$ for $k\neq 0$ is simply

$$\frac{1}{\log 2}\Gamma(\chi_k)\Big(\zeta\big(\chi_k,\tfrac{1}{4}\big)-\zeta\big(\chi_k,\tfrac{3}{4}\big)\Big)t^{-\chi_k}.$$

We can simplify, because

$$\zeta\big(\chi_k,\tfrac{3}{4}\big)=-\zeta\big(\chi_k,\tfrac{1}{4}\big).$$

Thus we obtain

$$C_n\sim[z^n]\frac{2}{t^3}\bigg(-\frac{1}{2}\log_2 t-\frac{5}{4}-\frac{\gamma}{2\log 2}+2\log_2\Gamma\big(\tfrac{1}{4}\big)-\log_2\pi \\ +\frac{2}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta\big(\chi_k,\tfrac{1}{4}\big)t^{-\chi_k}\bigg),$$

where, with $\varepsilon:=\sqrt{1-4z}$ and $t\sim 2\varepsilon$

$$C_n\sim[z^n]\bigg(-\frac{1}{8}\varepsilon^{-3}\log_2\varepsilon-\varepsilon^{-3}\Big(\frac{7}{16}+\frac{\gamma}{8\log 2}-\frac{1}{2}\log_2\Gamma\big(\tfrac{1}{4}\big)+\frac{1}{4}\log_2\pi\Big) \\ +\frac{1}{2\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta\big(\chi_k,\tfrac{1}{4}\big)\varepsilon^{-3-\chi_k}\bigg).$$

We find

$$[z^n]\varepsilon^{-3}\log\varepsilon\sim\frac{4^n\sqrt{n}}{\sqrt{\pi}}\Big(-\log n+(2-\gamma-2\log 2)\Big),$$

$$[z^n]\varepsilon^{-3}\sim\frac{2\cdot 4^n\sqrt{n}}{\sqrt{\pi}},$$

$$[z^n]\varepsilon^{-3-\chi_k}\sim 4^n\frac{n^{(1+\chi_k)/2}}{\Gamma\big(\frac{3+\chi_k}{2}\big)}.$$

By regrouping the preceeding results we obtain

$$C_n\sim\frac{4^n\sqrt{n}}{\sqrt{\pi}}\log n\cdot\frac{1}{8\log 2}$$

$$+\frac{4^n\sqrt{n}}{\sqrt{\pi}}\Big(-\frac{1}{4\log 2}-\frac{\gamma}{8\log 2}-\frac{5}{8}+\log_2\Gamma\big(\tfrac14\big)-\frac12\log_2\pi\Big)$$
$$+\frac12 4^n\sqrt{n}\frac{1}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)\zeta\big(\chi_k,\tfrac14\big)\frac{n^{\chi_k/2}}{\Gamma\big(\frac{3+\chi_k}{2}\big)}.$$

We have by the duplication formula [16]

$$\Gamma(\chi_k)\Big/\Gamma\Big(\frac{3+\chi_k}{2}\Big)=\frac{1}{1+\chi_k}\frac{1}{\sqrt{\pi}}\Gamma\Big(\frac{\chi_k}{2}\Big).$$

With

$$\binom{2n}{n}\sim\frac{4^n n^{-1/2}}{\sqrt{\pi}}$$

we find

$$\frac{D_n}{\binom{2n}{n}}\cdot\frac1n\sim\frac18\log_2 n$$
$$-\frac{1}{4\log 2}-\frac{\gamma}{8\log 2}-\frac{5}{8}+\log_2\Gamma\big(\tfrac14\big)-\frac12\log_2\pi$$
$$+\frac{1}{2\log 2}\sum_{k\neq 0}\zeta\big(\chi_k,\tfrac14\big)\frac{\Gamma(\chi_k/2)}{1+\chi_k}n^{\chi_k/2}.$$

Finally we have obtained the following asymptotic result.

Theorem 1. *The average number of exchanges in the odd-even merge of* $2n$ *elements satisfies*

$$B_n\sim\frac14 n\log_2 n+nB(\log_4 n),$$

where $B(x)$ *is a continuous periodic function of period 1; this function can be expanded as a Fourier series*

$$B(x)=\sum_{k\in\mathbb{Z}}b_k e^{2k\pi i x},$$

with

$$b_0=-\frac{1}{2\log 2}-\frac{\gamma}{4\log 2}-\frac34+2\log_2\Gamma\big(\tfrac14\big)-\log_2\pi$$

and for $k\neq 0$

$$b_k=\frac{1}{\log 2}\zeta\big(\chi_k,\tfrac14\big)\frac{\Gamma(\chi_k/2)}{1+\chi_k}.$$

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