Two proofs of Filipponi's formula for Lucas numbers of odd index

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In the recent paper [1], Filipponi presented the following formula without proof:

$$L_{2n+1} = \sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-1-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-2j}.$$

In this form, the term $j = \frac{n}{2}$ for even n has to be interpreted as a limit. However, we can avoid this trouble when we use the standard forms

$$G_n(w) = \sum_{0\le k\le n} \binom{n-k}{k} w^k.$$

They are studied very well in [2], and we have for $n \ge 0$

$$\sum_{0\le j\le \frac{n}{2}} (-1)^j \binom{n-1-j}{j} 3^{n-1-2j} \frac{4n-5j}{n-2j} = 3^n G_n(-\tfrac{1}{9}) + 3^{n-1} G_{n-1}(-\tfrac{1}{9}).$$

In exercise 7.34 of [2] we find also the generating function

$$G(z,w) = \sum_{n\ge 0} G_n(w) z^n = \frac{1}{1-z-wz^2}.$$

Hence

$$\begin{aligned}
&\sum_{n\ge 0} \Big(3^n G_n(-\tfrac{1}{9}) + 3^{n-1} G_{n-1}(-\tfrac{1}{9})\Big) z^n \\
&\quad = \sum_{n\ge 0} G_n(-\tfrac{1}{9})(3z)^n + z \sum_{n\ge 0} G_n(-\tfrac{1}{9})(3z)^n \\
&\quad = \frac{1+z}{1-3z+z^2}.
\end{aligned}$$

Since a trivial computation gives

$$\sum_{n\geq 0} L_{2n+1} z^n = \frac{1+z}{1-3z+z^2},$$

the proof is finished.

We can even get a general formula for L_{sn+t} with nonnegative integers $0 \leq t < s$; for that, we set up generating functions:

$$\begin{aligned} F_{s,t}(z) &= \sum_{n\geq 0} F_{sn+t} z^n = \alpha^t \sum_{n\geq 0} (\alpha^s z)^n + \beta^t \sum_{n\geq 0} (\beta^s z)^n = \alpha^t \frac{1}{1-\alpha^s z} + \beta^t \frac{1}{1-\beta^s z} \\ &= \frac{\alpha^t (1-\beta^s z) + \beta^t (1-\alpha^s z)}{(1-\alpha^s z)(1-\beta^s z)} = \frac{\alpha^t + \beta^t - ((-1)^t \alpha^{s-t} + (-1)^t \beta^{s-t}) z}{1-(\alpha^s+\beta^s) z + (-1)^s z} \\ &= \frac{L_t - (-1)^t L_{s-t} z}{1 - L_s z + (-1)^s z^2}. \end{aligned}$$

With the above defined function $G(z,w)$ this can be written as

$$F_{s,t}(z) = \big(L_t - (-1)^t L_{s-t}\big) G\left(L_s z, \frac{(-1)^{s-1}}{L_s^2}\right)$$

and therefore we get the formula

$$\begin{aligned} L_{sn+t} = L_t &\sum_{0\leq k\leq n} \binom{n-k}{k} (-1)^{(s-1)k} L_s^{n-2k} \\ &- (-1)^t L_{s-t} \sum_{0\leq k\leq n-1} \binom{n-1-k}{k} (-1)^{(s-1)k} L_s^{n-1-2k}. \end{aligned}$$

We don't know whether this formula is new, but it is easy to prove and generates infinitely many "Filipponi formulæ."

For the second (mechanical) proof, ("Zeilberger's algorithm") we note the following (see [3] and [2] for the underlying theory): Set $f(n,k) := (-1)^k \binom{n-1-k}{k} 3^{n-1-2k} \frac{4n-5k}{n-2k}$ and $F(n) := \sum_k f(n,k)$. Furthermore, set

$$g(n,k) := -\frac{9k(n-k)(4n-5k+5)}{(n-2k+2)(n-2k+1)(4n-5k)} f(n,k),$$

then

$$f(n+2,k) - 3f(n+1,k) + f(n,k) = g(n,k+1) - g(n,k)$$

(check!!), thus we get, on summing over k, $F(n+2) - 3F(n+1) + F(n) = 0$. Since the odd Lucas numbers also satisfy the recursion $L_{2n+5} - 3L_{2n+3} + L_{2n+1} = 0$ and two initial values match as well, the proof is finished.

References

[1] P. Filipponi. Combinatorial expressions for Lucas numbers. *The Fibonacci Quarterly*, 36:63–65, 1998.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics (Second Edition)*. Addison Wesley, 1994.

[3] M. Petkovšek, H. Wilf, and D. Zeilberger. $A = B$. A.K. Peters, Ltd., 1996.

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