RETAKH'S MOTZKIN PATHS AND SOME COMBINATORIAL COMMENTS

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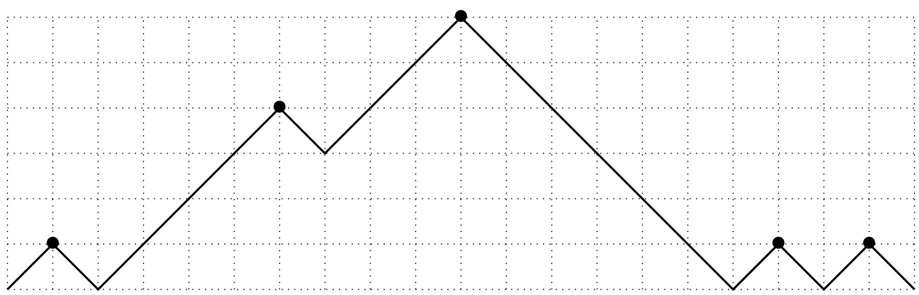
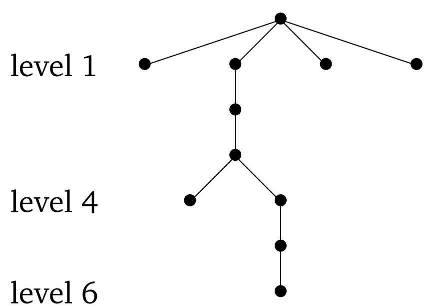
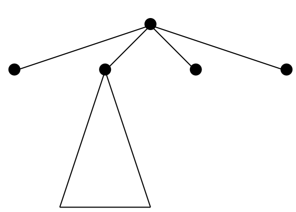
1. Introduction

V. Retakh introduced the following restricted class of Dyck paths: Peaks are only allowed on level 1 and on even-numbered levels. Here is an example, and the corresponding plane tree using the standard bijection.







Ekhad and Zeilberger [3] proved a few days ago that these restricted paths are enumerated by Motzkin numbers. Recall that the generating function of the Motzkin numbers $M(z)$ according to length satisfies $M = 1 + zM + z^2M^2$ and thus

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

In this note, I want to present a few additional observations, also including the height of the paths (or the associated plane trees). First, we are going to confirm the connection to Motzkin paths:

Since the level 1 is somewhat special, we consider only trees as symbolized by the triangle. We will use two generating functions, to deal with the odd/even situation. We have

$$G=\frac{z}{1-F} \quad \text{and} \quad F=\frac{zG}{1-G}.$$

Solvin, we find $G(z)=z^2M(z)$ and the total generating function is

$$\frac{z}{1-z}\sum_{r\ge0}\Big(\frac{G}{1-z}\Big)^r=zM(z),$$

as predicted. Recall that the number of nodes in trees is always one more than the half-length of the corresponding Dyck path.

We will compute the average height of such restricted paths, using singularity analysis of generating functions, as in [4, 5].

2. The height

Now we will use the substitution $z=\frac{v}{1+v+v^2}$, which occured for the first time in [8], but has been used more recently in different models where Motzkin numbers are involved [9, 7, 1], although within different models. For example, simply $M(z)=1+z+z^2$. We define

$$F_{k+1}=\frac{z}{1-\dfrac{zF_k}{1-F_k}}, \quad \text{with} \quad F_1=z.$$

There is a simple formula, viz.

$$F_k=\frac{v}{1+v}\frac{1-v^{2k}}{1-v^{2k+1}}.$$

This is easy to prove by induction. And then

$$G_k=\frac{zF_k}{1-F_k}=\frac{v^2}{1+v+v^2}\frac{1-v^{2k}}{1-v^{2k+2}}.$$

For $k\ge1$, G_k is the generating function of trees (like in the triangle) of height $\le 2k$.

Note that the height is currently counted in terms of nodes;

$$G_1=\frac{z^2}{1-z},$$

which describes a root with $l\ge1$ leaves attached to the root.

Now we incorporate the irregular beginning of the tree and compute

$$\frac{z}{1-z}\sum_{r\ge0}\Big(\frac{G_h}{1-z}\Big)^r=\frac{z}{1-z}\frac{1}{1-\frac{G_h}{1-z}}=v\frac{1-v^{2h+2}}{1-v^{2h+4}}.$$

From here onwards it seems to be more natural to define the height of the whole tree in terms of the number of *edges*, and then the quantity we just derived is the generating function of all trees with height $\le 2h$, for $h \ge 1$. Note that the limit $h \to \infty$ gives us simply $v = zM(z)$, which is consistent. There is also a contribution of trees of height ≤ 1, namely $\frac{z}{1-z} = \frac{v}{1+v^2}$, but this term is, when we compute the average height, irrelevant and only contributes to the error term, as we only compute only the leading term, which is of order $\sqrt{n}$.

So, apart from normalization, we are led to investigate

$$\begin{aligned}\sum_{h\ge1}2h\Big[v\frac{1-v^{2h+2}}{1-v^{2h+4}}-v\frac{1-v^{2h}}{1-v^{2h+2}}\Big]&=2v(1-v^{-2})\sum_{h\ge1}h\Big[\frac{v^{2h+4}}{1-v^{2h+4}}-\frac{v^{2h+2}}{1-v^{2h+2}}\Big]\\&=2v(1-v^{-2})\sum_{h\ge0}h\frac{v^{2h+4}}{1-v^{2h+4}}-2v(1-v^{-2})\sum_{h\ge0}(h+1)\frac{v^{2h+4}}{1-v^{2h+4}}\\&=-2v+\frac{2(1-v^2)}{v}\sum_{h\ge1}\frac{v^{2h}}{1-v^{2h}}.\end{aligned}$$

Note that we could get explicit coefficients form here, using trinomial coefficient, $\binom{n,3}{k} = [v^k](1+v+v^2)^n$ (notation from [2]) This has to be expanded around $v = 1$, which is a standard application of the Mellin transform. Details are worked out in [6], for example:

$$\sum_{h\ge1}\frac{v^{2h}}{1-v^{2h}}=\sum_{k\ge1}d(k)v^{2k}\sim-\frac{\log(1-v^2)}{1-v^2}\sim-\frac{\log(1-v)}{2(1-v)}.$$

Note that $d(k)$ is the number of divisors of k. Consequently

$$-2v+\frac{2(1-v^2)}{v}\sum_{h\ge1}\frac{v^{2h}}{1-v^{2h}}=\sim-2\log(1-v).$$

We have $1-v\sim\sqrt{3}\sqrt{1-3z}$, and $z=\frac13$ is the relevant singularity when discussing Motzkin numbers. We can continue

$$-2v+\frac{2(1-v^2)}{v}\sum_{h\ge1}\frac{v^{2h}}{1-v^{2h}}\sim-\log(1-3z).$$

The coefficient of z^n in this is $\frac{3^n}{n}$. This has to be divided by

$$[z^n]zM(z)=[z^{n-1}]M(z)\sim\frac{3^{n+\frac12}}{2\sqrt{\pi}n^{3/2}},$$

with the final result for the average height of the restricted Dyck paths:

$$\sim 2\sqrt{\frac{\pi n}{3}}.$$

Recall [8] that the average height of Motzkin paths of length n is asymptotic to

$$\sqrt{\frac{\pi n}{3}}.$$

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