ON THE NUMBER OF PARTITIONS OF $\{1, \cdots, n\}$ INTO r SETS OF EQUAL CARDINALITIES AND EQUAL SUMS

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In [2] it is proved that the number $A(n)$ of ways to partition the set $\{1, 2,\cdots, n\}$ into two sets of equal cardinalities and equal sums of elements is asymptotically given by

$$A(n) \sim \frac{2^n}{n^2}\frac{4\sqrt{3}}{\pi}, \qquad n\to\infty, \quad n\equiv 0 \pmod 4. \tag{1}$$

The aim of this note is to extend this result to the case of r subsets. Our result is

THEOREM. *The number $A_r(n)$ of ways to partition the set $\{1, 2,\cdots, n\}$ into r sets of equal cardinalities and equal sums of elements is given by*

$$A_r(n) \sim \frac{2\sqrt{3}}{(2\pi)^{r-1}}\cdot\frac{r^r r^n}{n^r}, \qquad n\to\infty, \qquad \begin{cases} n\equiv 0 \pmod r & \text{if } r \text{ odd} \\ n\equiv 0 \pmod{2r} & \text{if } r \text{ even}. \end{cases} \tag{2}$$

Instead of giving an exact proof which would be rather lengthy (partially because of a necessarily clumsy notation) we just stress the main ideas of the proof and refer to [2] for a full treatment in the case $r=2$.

If f is a power series in variables $x_1,\cdots, x_s$, $\langle x_1^{i_1}\cdots x_s^{i_s}\rangle f$ will denote the coefficient of $x_1^{i_1}\cdots x_s^{i_s}$ in f. It is now quite clear that $A_r(n)$ is given by

$$\langle (x_2\cdots x_r)^{n/r}(z_2\cdots z_r)^{n(n+1)/2r}\rangle \prod_{k=1}^{n}(1+x_2z_2^k+\cdots+x_rz_r^k)\,; \tag{3}$$

to derive the asymptotics of $A_r(n)$ we want to use Cauchy's Theorem, so more-dimensional integrals come into play. From (3) we easily see that $A_r(n) \neq 0$ provided

$$r \equiv 1 \pmod 2, \quad n \equiv 0 \pmod r \qquad \text{or} \tag{4}$$
$$r \equiv 0 \pmod 2, \quad n \equiv 0 \pmod{2r},$$

which is assumed to hold throughout this paper. In the sequel we need some preliminaries.

Let I be the matrix of unity, and J the matrix whose entries are all equal to 1, both of dimension $(r-1) \times (r-1)$. It is easy to establish the following results:

$$\det(rI - J) = r^{r-2}, \tag{5}$$

$$\det\left(n\left(I - \frac{1}{r}J\right)\right) = n^{r-1} \cdot \frac{1}{r}. \tag{6}$$

Let

$$A = \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix},$$

then

$$\det A = (\det B)^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}. \tag{7}$$

We use some shorthand notations:

For $\sum a_i$ we write $[a]$, for $\sum a_i b_i$ we write $[ab]$ and so on.

We use the substitutions

$$x_j = e^{it_j}, \qquad z_j = e^{is_j}, \tag{8}$$

then

$$A_r(n) = \frac{1}{(2\pi)^{2(r-1)}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{k=1}^{n} (1 + e^{i(t_2+ks_2)} + \cdots + e^{i(t_r+ks_r)}) \times e^{-i(n/r)[t]} e^{-i(n(n+1)/2r)[s]} dt_2 \cdots dt_r\, ds_2 \cdots ds_r. \tag{9}$$

For asymptotical purposes we may replace

$$(1 + e^{ix_2} + \cdots + e^{ix_r}) \quad \text{by} \quad r\left(1 + \frac{i}{r}[x] - \frac{1}{2r}[x^2]\right)$$

and thus

$$(1 + e^{ix_2} + \cdots + e^{ix_r})\, e^{-(i/r)[x]} \quad \text{by} \quad r\, e^{-(1/2r)[x^2] + (1/2r^2)[x]^2}. \tag{10}$$

Applying (10) to (9) we have an asymptotic equivalent for $A_r(n)$; in this formula the integrals may be replaced by $\int_{-\infty}^{\infty}$. So we find

$$\begin{aligned} &A_r(n)\, r^{-n}(2\pi)^{2(r-1)} \\ &\sim \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2r}\sum_{k=1}^{n}[(t+ks)^2] + \frac{1}{2r^2}\sum_{k=1}^{n}[t+ks]^2\right) \\ &\qquad \times dt_2\cdots dt_r\, ds_2\cdots ds_r\,. \end{aligned} \tag{11}$$

Now

$$\begin{aligned} &\sum_{k=1}^{n}[(t+ks)^2] \\ &\quad = \sum_{k=1}^{n}[t^2+2kts+k^2s^2] \sim n[t^2] + n^2[ts] + \frac{n^3}{3}[s^2]\,, \end{aligned} \tag{12}$$

and similarily

$$\sum_{k=1}^{n}[t+ks]^2 \sim n[t]^2 + n^2[t][s] + \frac{n^3}{3}[s]^2\,. \tag{13}$$

To compute the integral we have to write

$$-\frac{1}{2r}\left(n[t^2] + n^2[ts] + \frac{n^3}{3}[s^2]\right) + \frac{1}{2r^2}\left(n[t]^2 + n^2[t][s] + \frac{n^3}{3}[s]^2\right)$$

as

$$-\frac{1}{2}[t_2,\cdots,t_r;\ s_2,\cdots,s_r]\cdot A\cdot[t_2,\cdots,t_r;\ s_2,\cdots,s_r]^t \tag{14}$$

with a certain matrix A of dimension $(2(r-1))\times(2(r-1))$. It is quite easy to see that

$$\begin{aligned} A &= \frac{n}{r}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix} + \frac{n^2}{2r}\begin{bmatrix} 0 & I\\ I & 0\end{bmatrix} + \frac{n^3}{3r}\begin{bmatrix} 0 & 0\\ 0 & I\end{bmatrix} \\ &\qquad - \frac{n}{r^2}\begin{bmatrix} J & 0\\ 0 & 0\end{bmatrix} - \frac{n^2}{2r^2}\begin{bmatrix} 0 & J\\ J & 0\end{bmatrix} - \frac{n^3}{3r^2}\begin{bmatrix} 0 & 0\\ 0 & J\end{bmatrix} \\ &= \frac{n}{r^2}\begin{bmatrix} aB & bB\\ cB & dB\end{bmatrix}, \end{aligned} \tag{15}$$

with $B = rI - J$, $a = 1$, $b = c = n/2$, $d = n^2/3$. So

$$\det A = \left(\frac{n}{r^2}\right)^{2(r-1)} \det{}^2[rI - J]\cdot\begin{vmatrix} 1 & n/2\\ n/2 & n^2/3\end{vmatrix} = \frac{1}{12}\left(\frac{n}{r}\right)^{2r}. \tag{16}$$

It is well known [1] that

$$\begin{aligned}(17)\qquad &\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}[t_2,\cdots,s_r]A[t_2,\cdots,s_r]^t\right)dt_2\cdots ds_r\\ &=(\det A)^{-1/2}\times(\sqrt{2\pi})^{2(r-1)}\\ &=\frac{(2\pi)^{r-1}2\sqrt{3}\,r^r}{n^r},\end{aligned}$$

which leads to (2).

References

[1] N. L. JOHNSON, S. KOTZ, *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley & Sons, New York, 1972.

[2] H. PRODINGER, *On the number of partitions of* $\{1, \cdots, n\}$ *into two sets of equal cardinalities and equal sums*, Canad. Math. Bull., 25 (1982), 238–241.

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