SUMS OF CHOI, ZÖRNIG, AND RATHIE — AN ELEMENTARY APPROACH

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Abstract. The sums in the title and any number of similar ones are obtained in a completely elementary and simple way.

The sum

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}$$

has gained a fair amount of attraction, see [1] and the references given therein.

We study here the slightly more general sum

$$S(n,m):=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2^k(m+k)}$$

by *completely elementary tools.* ($n\in\mathbb{N}_0$, $m\in\mathbb{N}$.)

There is the alternative formula

$$S(n,m)=\frac{n!(m-1)!}{2^n(n+m)!}\sum_{k=0}^{n}\binom{m+n}{k},$$

which can be proved using *Pfaff's reflection law*, see [2]. But it can also be proved by an *induction* on n, which is a simple exercise left to the reader. It is amusing that this sum (and its alternative) appear also in [3].

This alternative formula is particularly useful if m is close to zero or close to n, since then the sum can be evaluated in closed form. To wit, let $m=n+d$, with $d\in\mathbb{N}_0$, then

$$S(n,n+d)=\frac{n!(n+d-1)!}{2^n(2n+d)!}\left[2^{2n+d-1}-\frac{1}{2}\sum_{k=n+1}^{n+d-1}\binom{2n+d}{k}\right].$$

Note that for $d=0$, the expression in the bracket must be interpreted as $2^{2n-1}+\frac{1}{2}\binom{2n}{n}$, which is consistent (see [2]).

So we evaluated the sum mentioned in the beginning:

$$S(n-1,n+1)=\frac{(n-1)!n!}{2^{n-1}(2n)!}\left[2^{2n-1}-\frac{1}{2}\binom{2n}{n}\right]=\frac{(n-1)!n!2^n}{(2n)!}-\frac{1}{n2^n}.$$

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The paper [1] contains two main results ((1.11) and (1.12) loc. cit.), which go like this:

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k}{2^k(n+k)(n+k+1)}\\
&\qquad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{2^k}\left[\frac{n+1}{n+1+k}-\frac{n}{n+k}\right]\\
&\qquad=(n+1)S(n,n+1)-nS(n,n)\\
&\qquad=(n+1)\frac{n!n!}{(2n+1)!}2^n-n\left[\frac{n!(n-1)!}{(2n)!}2^{n-1}+\frac{1}{n2^{n+1}}\right]\\
&\qquad=\frac{n!n!}{(2n+1)!}2^{n-1}-\frac{1}{2^{n+1}};
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{k}{2^k(n+k)(n+k+1)}\\
&\qquad=\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{2^k}\left[\frac{n+1}{n+1+k}-\frac{n}{n+k}\right]\\
&\qquad=(n+1)S(n-2,n+1)-nS(n-2,n)\\
&\qquad=(n+1)\frac{(n-2)!n!}{2^{n-2}(2n-1)!}\left[2^{2n-2}-\frac{1}{2}\sum_{k=n-1}^{n}\binom{2n-1}{k}\right]\\
&\qquad\quad-n\frac{(n-2)!(n-1)!}{2^{n-2}(2n-2)!}\left[2^{2n-3}-\frac{1}{2}\sum_{k=n-1}^{n-1}\binom{2n-2}{k}\right]\\
&\qquad=\frac{3n!n!2^n}{(n-1)(2n)!}-\frac{n+2}{(n-1)2^{n-1}}.
\end{aligned}$$

It is easy to generate any number of similar examples — preferably with a computer!

References

[1] J. Choi, P. Zörnig, and A.K. Rathie. Sums of certain classes of series. *Comm. Korean Math. Soc.*, 14:641–647, 1999.

[2] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics (Second Edition)*. Addison Wesley, 1994.

[3] A. Knopfmacher and H. Prodinger. A simple card guessing game revisited. *Electron. J. Combin.*, 8(2):Research Paper 13, 9 pp. (electronic), 2001. In honor of Aviezri Fraenkel on the occasion of his 70th birthday.

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