

GENERATING FUNCTIONS RELATED TO PARTITION FORMULÆ FOR FIBONACCI NUMBERS

HELMUT PRODINGER

ABSTRACT. The generating functions of 2 double schemes of numbers are explicitly computed using the kernel method, which leads to easy verification proofs of partition formulæ for Fibonacci numbers.

1. INTRODUCTION

The numbers given by the recursions

$$\begin{aligned}b_{n+1,k} &= c_{n,k-1} + 2c_{n,k} - b_{n,k}, \\c_{n+1,k} &= b_{n+1,k} + 2b_{n+1,k+1} - c_{n,k}\end{aligned}$$

for $n \geq 0$, with $b_{0,0} = c_{0,0} = 1$, $c_{n,-1} = c_{n,0}$ are used in [1] to partition the Fibonacci numbers. We want to shed new light on these numbers, by computing their (bivariate) generating functions. These lead then also to straight forward verification proofs of the partition formulæ given in [1]

2. THE GENERATING FUNCTIONS

Introducing generating functions

$$\begin{aligned}B(z, x) &:= \sum_{0 \leq k \leq n} b_{n,k} z^n x^k, \\C(z, x) &:= \sum_{0 \leq k \leq n} c_{n,k} z^n x^k,\end{aligned}$$

these recursions translate into

$$\begin{aligned}B(z, x) &= 1 + zx C(z, x) + 2z C(z, x) - z B(z, x) + z C(z, 0), \\C(z, x) &= B(z, x) + \frac{2}{x} [B(z, x) - B(z, 0)] - z C(z, x).\end{aligned}$$

This leads to

$$C(z, x) = -\frac{x + z(x - 4)C(z, 0)}{zx^2 - z^2x + 2zx - x + 4z}.$$

To solve that, we factor the denominator:

$$C(z, x) = \frac{-x + z(4 - x)C(z, 0)}{z(x - r_1(z))(x - r_2(z))}$$

Date: February 29, 2008.

with

$$r_{1,2}(z) = \frac{(1-z)^2 \pm (1+z)\sqrt{1-6z+z^2}}{2z}.$$

Since $1/(x - r_2(z))$ has no power series expansion in z and x , the factor must cancel, i.e.,

$$-r_2(z) + z(4 - r_2(z))C(z, 0) = 0,$$

whence

$$C(z, 0) = \frac{-1 + 4z - z^2 + (1+z)\sqrt{1-6z+z^2}}{2z(1-7z+z^2)}.$$

This is the famous *kernel method*, see, e.g., [2].

After cancellation, this leads to

$$C(z, x) = \frac{4r_2(z)}{2z(1-4xr_2(z))} \frac{1-10z+z^2+(1+z)\sqrt{1-6z+z^2}}{1-7z+z^2},$$

and

$$[x^k]C(z, x) = \frac{(4r_2(z))^{k+1}}{2z} \frac{1-10z+z^2+(1+z)\sqrt{1-6z+z^2}}{1-7z+z^2}.$$

From this we get

$$B(z, x) = \frac{-(1+z)(z^2x - 8x + 4z + x) + (-z^2x + 12z + 4zx - x)\sqrt{1-6z+z^2}}{2(1-7z+z^2)(-z^2x + 2zx - x + 4z + zx^2)}.$$

The formula

$$f_{4(n+1)} = 3 \sum_{k \geq 0} 4^k c_{n,k},$$

given in [1], can now easily be verified, since the generating function of the righthandside is

$$3C(z, 4) = \frac{3}{1-7z+z^2},$$

which is also the generating function of the lefthandside, which can be seen for example from the Binet form of the Fibonacci numbers.

The other formula

$$f_{4n+2} = b_{n,0} + \frac{3}{2} \sum_{k \geq 1} 4^k b_{n,k},$$

follows from the generating function

$$B(z, 0) + \frac{3}{2} (B(z, 4) - B(z, 0)) = \frac{1+z}{1-7z+z^2}.$$

REFERENCES

- [1] Ph. Fahr and C.-M. Ringel, A Partition Formula for Fibonacci Numbers, *Journal of Integer Sequences* **11** (2008), Article 08.1.4 (9 pages).
- [2] H. Prodinger, The Kernel Method: A Collection of Examples, *Séminaire Lotharingien de Combinatoire*, B50f (2004), 19 pp.

HELMUT PRODINGER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STELLENBOSCH, 7602 STELLENBOSCH, SOUTH AFRICA

E-mail address: hproding@sun.ac.za