WORDS WITH A GENERALIZED RESTRICTED GROWTH PROPERTY

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Dedicated to the memory of N. G. de Bruijn

ABSTRACT. Words where each new letter (natural number) can never be too large, compared to the ones that were seen already, are enumerated. The letters follow the geometric distribution. Also, the maximal letter in such words is studied. The asymptotic answers involve small periodic oscillations. The methods include a chain of techniques: exponential generating function, Poisson generating function, Mellin transform, depoissonization.

1. INTRODUCTION

When Knuth started his fundamental series of books *The Art of Computer Programming* [8], de Bruijn was (one of) his asymptotic advisor(s). In particular, he suggested how to evaluate sums like

$$\sum_{k\ge1}\Bigl(1-e^{-n/2^k}\Bigr)\quad\text{and}\quad\sum_{k\ge1}d(k)e^{-k^2/n},$$

where $d(k)$ is the number of divisors of k. Although the word was not mentioned in the first editions, in essence is was the *Mellin transform* that found its way into [9]. Around the same time, the paper [2] appeared, which has 158 citations by google scholar. This paper has a third coauthor, S. O. Rice, who also suggested asymptotic methods to Knuth; there is an innocent exercise in [9], which lead later to developments called *Rice's method*, see [4].

We briefly review the Mellin transform method in asymptotic enumeration, compare with [3, 5].

$$\mathscr{M}[f(x);s]=f^*(s)=\int_0^\infty f(x)x^{s-1}dx.$$

There is the *harmonic sum property*

$$\mathscr{M}\Bigl[\sum_{k\ge1}a_kf(b_kx);s\Bigr]=\sum_{k\ge1}a_kb_k^{-s}\cdot\mathscr{M}[f(x);s]. \tag{1}$$

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This is particularly useful if the series has a closed form evaluation (often in terms of the zeta function etc.).

Typically the Mellin transform exists in a *vertical strip* of the complex plane. There is an *inversion formula*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds,$$

where c must be in the vertical strip. Shifting the line of integration to the left/right and collecting residues provides the asymptotic expansion. The choice of left/right depends on whether one needs the expansion for $x \to \infty$ or $x \to 0$; see the converse mapping theorem in [3] for a precise statement of this fact.

The most prominent example is $f(x) = e^{-x}$, so that $f^*(s) = \Gamma(s)$, whence the term *Gamma function method* was originally coined. During the last 40 years, de Bruijn's suggestion led to numerous further developments and applications.

In the technical part of this paper, we will indeed use the Mellin transform to deal with a combinatorial (discrete probability) problem. As often in combinatorics, the problem is not difficult to describe, although the solution requires some technical machinery. We consider words $w_1w_2\dots w_n$ where the letters are positive integers, and integer k appears with (geometric) probability pq^{k-1}, and $p+q=1$. The letters are independent from each other. The *restricted growth property* is satisfied when

$$w_k \le 1 + \max\{w_1,\dots,w_{k-1}\} \quad \text{for all } k \quad \text{and } w_0 = 0.$$

The words that satisfy the restricted growth property are related to *set partitions* and *approximate counting* [10, 12]. The asymptotic enumeration of restricted words of length n and the asymptotic study of $\max\{w_1,\dots,w_n\}$ was done in [11, 12] using the above mentioned Rice method.

Now it is a natural extension to introduce a parameter:

$$w_k \le d + \max\{w_1,\dots,w_{k-1}\} \quad \text{for all } k \quad \text{and } w_0 = 0. \tag{2}$$

For $d \ge 2$, the asymptotic problems are of a more delicate nature, and that is what we will do here. Rice's method is based on explicit enumerations represented as *alternating sums*. We use here a combination of techniques that is more flexible: poissonization/depoissonization and Mellin transform. Poissonization is the process of replacing the fixed n by a random variable which is Poisson distributed with parameter z; depoissonization is the reversed process that allows to go back from z to n. Typically, if $f(z)$ is an exponential generating function of a sequence a_n, then, with $\tilde{f}(z) := e^{-z}f(z)$, $a_n \sim \tilde{f}(n)$, provided certain conditions are satisfied. The asymptotic study of the behaviour of the poissonized version when $z \to \infty$ is achieved using the Mellin transform. This is the rough plan; the details are in the following sections.

Notation. We collect here some notation which we are going to use throughout this work. First,

$$Q := 1/q, \qquad L := \log Q, \qquad \chi_k := 2k\pi i/L,\ k \in \mathbb{Z}.$$

Moreover, if $f(z)$ is a meromorphic function with singularity at $z = \rho$ and singularity expansion $E(z)$, then we will write $f(z) \asymp E(z)$.

2. Asymptotic enumeration of words satisfying the restricted growth property

Let p_n be the probability that a random word $w_1 \dots w_n$ satisfies the restricted growth property (2).

We first condition on whether w_1 is l with $1 \le l \le d$. Note that the probability for this is pq^{l-1} and the probability that any other letter is $\le l$ is

$$pq + pq^2 + \cdots + pq^{l-1} = p\frac{1-q^l}{1-q} = 1 - q^l.$$

Hence, by further conditioning on the number of letters $\le l$ in $w_2 \dots w_{n+1}$, we obtain

$$p_{n+1} = \sum_{l=1}^{d} pq^{l-1} \sum_{j=0}^{n} \binom{n}{j} (1-q^l)^{n-j} q^{lj} p_j \qquad (n \ge 0)$$

with initial condition $p_0 = 1$.

The binomial convolution on the right-hand side of the recurrence above suggests the use of *exponential generating functions*. Therefore, set

$$f(z) := \sum_{n \ge 0} p_n \frac{z^n}{n!}.$$

Then

$$f'(z) = \sum_{l=1}^{d} pq^{l-1} e^{(1-q^l)z} f(q^l z)$$

Next, consider the *Poisson generating function*

$$\tilde{f}(z) := e^{-z} \sum_{n \ge 0} p_n \frac{z^n}{n!}.$$

Then, the above differential-functional equation becomes

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^{d} pq^{l-1} \tilde{f}(q^l z).$$

The goal is now to find the behaviour of $\tilde{f}(z)$ as $z \to \infty$, since $p_n \sim \tilde{f}(n)$ by general principles. This goal will be achieved using the Mellin transform. Recall that the Mellin transform of a derivative is given by [3]

$$\mathscr{M}[\tilde{f}(z); \omega] = -(\omega - 1)\mathscr{M}[\tilde{f}(z); \omega - 1].$$

Using this and (1) from the introduction yields

$$\mathscr{M}[\tilde{f}(z); \omega] - (\omega - 1)\mathscr{M}[\tilde{f}(z); \omega - 1] = \sum_{l=1}^{d} pq^{l-1-l\omega} \cdot \mathscr{M}[\tilde{f}(z); \omega].$$

This functional equation can be simplified by using the Gamma function as normalization factor. Therefore, define

$$\bar{\mathscr{M}}[\tilde{f}(z);\omega]=\frac{\mathscr{M}[\tilde{f}(z);\omega]}{\Gamma(\omega)}.$$

Then

$$\bar{\mathscr{M}}[\tilde{f}(z);\omega]=\frac{\bar{\mathscr{M}}[\tilde{f}(z);\omega-1]}{P(q^{-\omega})},$$

where $P(z)=1-p\sum_{l=1}^{d}q^{l-1}z^{l}$.

Next, observe that the above functional equation has the general solution

$$\bar{F}(\omega):=\bar{\mathscr{M}}[\tilde{f}(z);\omega]=\frac{c}{P(q^{-\omega})\Omega(q^{-\omega})}, \tag{3}$$

where $\Omega(s)=\prod_{j\geq 1}P(sq^{j})$. Note that for $d=1$, we have

$$\Omega(s)=\prod_{j\geq 1}(1-pq^{j}s)=Q(ps)=(pqs;q)_{\infty}.$$

The notation $Q(s)$ is often used in Computer Science contexts [6], whereas $(pqs;q)_{\infty}$ is used in q-hypergeometric functions [1].

Next, we need to find c in (3). Therefore, observe that from $\tilde{f}(0)=1$ and the direct mapping theorem from [3], we have, as $z\to 0$,

$$\mathscr{M}[\tilde{f}(z);\omega]\asymp\frac{1}{\omega}.$$

Consequently,

$$\lim_{\omega\to 0}\bar{\mathscr{M}}[\tilde{f}(z);\omega]=\lim_{\omega\to 0}\frac{\mathscr{M}[\tilde{f}(z);\omega]}{\Gamma(\omega)}=\lim_{\omega\to 0}\frac{1/\omega+\cdots}{1/\omega+\cdots}=1.$$

Hence, by taking the limit as $\omega\to 0$ in (3)

$$c=P(1)\Omega(1)=q^{d}\Omega(1).$$

Summarizing,

$$\mathscr{M}[\tilde{f}(z);\omega]=\frac{q^{d}\Omega(1)\Gamma(\omega)}{P(q^{-\omega})\Omega(q^{-\omega})}. \tag{4}$$

Now that this function is known, we continue our program and (applying the inversion formula) collect residues. In order to identify them, we need the following technical lemma.

Lemma 1. *Let ρ denote the unique positive root of $P(z)$. Then, ρ is simple, $\rho>1$ and the only root with $|z|\leq\rho$.*

Proof. Obviously, there exists a unique positive root ρ which is simple. Moreover, since

$$P(1)=1-p\sum_{l=1}^{d}q^{l-1}=q^{d}>0,$$

we have that $\rho > 1$. Next, observe that for $|z| \leq \rho$

$$\left|p\sum_{l=1}^{d} q^{l-1}z^l\right| \leq p\sum_{l=1}^{d} q^{l-1}|z|^l \leq p\sum_{l=1}^{d} q^{l-1}\rho^l = 1.$$

If $|z| < \rho$, then the last inequality is strict; if $|z| = \rho$, then in order that z is a zero of $P(z)$, we must have equality in the triangle inequality which is only possible if z is on the positive real line. ∎

From this lemma, we know that (4) has poles at $\log_Q \rho + \chi_k$ with singularity expansion

$$\mathscr{M}[\tilde{f}(z);\omega] \asymp \frac{q^d\Omega(1)\Gamma(\log_Q \rho + \chi_k)}{L\rho P'(\rho)\Omega(\rho)(\omega - \log_Q \rho - \chi_k)}.$$

Inverse Mellin transform then yields, as $z \to \infty$,

$$\tilde{f}(z) \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)} z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k}.$$

The last step is depoissonization. Here, we use the notation of JS-admissibility defined in Definition 1 of Section 2.3 in [6]. (The letters J and S are used to honour the authors of the early effort [7].) The following lemma is sufficient for our purpose.

Lemma 2. *Let $\tilde{f}(z)$ and $\tilde{g}(z)$ be entire functions with*

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^{d} pq^{l-1}\tilde{f}(q^l z) + \tilde{g}(z).$$

Then

$$\tilde{f}(z) \text{ is JS-admissible} \quad \Longleftrightarrow \quad \tilde{g}(z) \text{ is JS-admissible.}$$

Proof. Similar as Proposition 2.4 in [6] (only minor modifications are necessary). ∎

From this result, we obtain that $\tilde{f}(z)$ is JS-admissible (since $\tilde{g}(z) = 0$ which is clearly JS-admissible). Consequently, by Proposition 2.2 in [6],

$$p_n \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)} n^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) n^{-\chi_k}.$$

We summarize this result in the following theorem.

Theorem 1. *The probability p_n that a random word of length n satisfies the restricted growth property is asymptotically given by*

$$p_n \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}\Gamma(\log_Q \rho) n^{-\log_Q \rho} + n^{-\log_Q \rho}\Psi(\log_Q n),$$

where $\Psi(z)$ is a 1-periodic function with average value equal to zero and Fourier series

$$\Psi(z) = -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)} \sum_{k\neq 0} \Gamma(\log_Q \rho + \chi_k) e^{-2k\pi i}.$$

Remark 1. As often in the analysis of algorithms, we have small periodic oscillations (smallness comes from the exponential decay of the Gamma function along vertical lines).

For $d = 1$, we have $\rho = 1/p$, and, ignoring the oscillating part, we have

$$p_n \sim \frac{qQ(p)}{LQ(1)} \Gamma(-\log_Q p) n^{\log_Q p}.$$

This matches a formula given earlier in [11, 12].

3. The maximal letter in restricted words

The random variable X_n as indicated in the title of this section is reminiscent of the *height* of planar (=planted plane) trees, as studied by de Bruijn, Knuth, and Rice [2].

Our goal here is to find the expected value of X_n which is given by

$$\mathbb{E}(X_n) = \frac{\sum_k k p_{n,k}}{p_n}, \tag{5}$$

where $p_{n,k}$ is the probability that a random word $w_1 \dots w_n$ satisfying (2) has largest letter equal to k.

We will first derive a recurrence for $p_{n,k}$. Therefore, we use the same argument as in the previous section. This yields

$$p_{n+1,k} = \sum_{l=1}^{d} pq^{l-1} \sum_{j=0}^{n} \binom{n}{j} (1-q^l)^{n-j} q^{lj} p_{j,k-l} \qquad (n \ge 0;\ k \ge 1)$$

with initial conditions $p_{n,0} = [\![n = 0]\!]$, $p_{0,k} = [\![k = 0]\!]$ and $p_{j,k} = 0$ for $k < 0$. Now, again consider the Poisson generating function

$$\tilde{H}(z, u) := e^{-z} \sum_{n,k \ge 0} p_{n,k} u^k \frac{z^n}{n!}.$$

Then

$$\tilde{H}(z,u) + \frac{\partial}{\partial z} \tilde{H}(z,u) = \sum_{l=1}^{d} pq^{l-1} u^l \tilde{H}(q^l z, u). \tag{6}$$

Next, in order to compute the expectation of X_n, set

$$\tilde{h}(z) = \frac{\partial}{\partial u} \tilde{H}(z,u)\Big|_{u=1}.$$

Note that this is the Poisson generating function of the numerator of (5). Differentiating (6) with respect to u and setting $u = 1$ gives

$$\tilde{h}(z) + \tilde{h}'(z) = \sum_{l=1}^{d} pq^{l-1} \tilde{h}(q^l z) + \tilde{g}(z)$$

with

$$\tilde{g}(z) = \sum_{l=1}^{d} lpq^{l-1}\tilde{f}(q^l z).$$

We again apply Mellin transform to this differential-functional equation. This yields

$$\mathscr{M}[\tilde{h}(z);\omega] - (\omega-1)\mathscr{M}[\tilde{h}(z);\omega-1] = \left(1-P(q^{-\omega})\right)\mathscr{M}[\tilde{h}(z);\omega] + \mathscr{M}[\tilde{g}(z);\omega].$$

Next, set

$$\bar{\mathscr{M}}[\tilde{h}(z);\omega] = \frac{\mathscr{M}[\tilde{h}(z);\omega]}{\Gamma(\omega)}$$

and

$$\frac{\mathscr{M}[\tilde{g}(z);\omega]}{\Gamma(\omega)} = -q^{-\omega}P'(q^{-\omega})\bar{F}(\omega).$$

Then

$$\bar{\mathscr{M}}[\tilde{h}(z);\omega] = \frac{\bar{\mathscr{M}}[\tilde{h}(z);\omega-1]}{P(q^{-\omega})} - \frac{q^{-\omega}P'(q^{-\omega})\bar{F}(\omega)}{P(q^{-\omega})}.$$

The solution of this recurrence is given by

$$\bar{\mathscr{M}}[\tilde{h}(z);\omega] = -\sum_{l\geq 0}\frac{\Omega(q^{-\omega+l})q^{-\omega+l}P'(q^{-\omega+l})\bar{F}(\omega-l)}{P(q^{-\omega})\Omega(q^{-\omega})} + \frac{c}{P(q^{-\omega})\Omega(q^{-\omega})}.$$

where, from $\tilde{h}(0)=0$ and letting $w\to 0$ as in the previous section gives

$$c = -q^d\Omega(1)\alpha_p, \qquad \alpha_p := -\sum_{l\geq 0}\frac{q^lP'(q^l)}{P(q^l)}.$$

Overall,

$$\begin{aligned}\mathscr{M}[\tilde{h}(z);\omega] = &-\frac{q^d\Omega(1)q^{-\omega}P'(q^{-\omega})\Gamma(\omega)}{P(q^{-\omega})^2\Omega(q^{-\omega})}\\ &-\Gamma(\omega)\left(\sum_{l\geq 1}\frac{\Omega(q^{-\omega+l})q^{-\omega+l}P'(q^{-\omega+l})\bar{F}(\omega-l)}{P(q^{-\omega})\Omega(q^{-\omega})} + \frac{q^d\Omega(1)\alpha_p}{P(q^{-\omega})\Omega(q^{-\omega})}\right). \quad (7)\end{aligned}$$

As before, we will use inverse Mellin transform and collect residues. Therefore, we treat the two terms on the right-hand side of (7) separately. First, for the first one, we have double poles at $c := \log_Q \rho + \chi_k$. For the singularity expansion at c note that

$$P(q^{-\omega}) = L\rho P'(\rho)(\omega-c) + \frac{L^2\rho P'(\rho) + L^2\rho^2P''(\rho)}{2}(\omega-c)^2 + \cdots$$

and

$$q^{-\omega}P'(q^{-\omega}) = \rho P'(\rho) + (L\rho P'(\rho) + L\rho^2P''(\rho))(\omega-c) + \cdots.$$

From this,

$$\frac{q^{-\omega}P'(q^{-\omega})}{P(q^{-\omega})^2} = \frac{1}{L^2\rho P'(\rho)(\omega-c)^2} + d^* + \cdots,$$

where d^* is a constant. Next, we need

$$\frac{1}{\Omega(q^{-\omega})} = \frac{1}{\Omega(\rho)}\left(1 + L\alpha(\omega - c) + \cdots\right),$$

where

$$\alpha = -\sum_{l\geq 1} \frac{\rho q^l P'(\rho q^l)}{P(\rho q^l)}.$$

Plugging this into the first term and using $\Gamma(\omega) = \Gamma(c) + \Gamma'(c)(\omega - c) + \cdots$, we obtain

$$-\frac{q^d\Omega(1)q^{-\omega}P'(q^{-\omega})\Gamma(\omega)}{P(q^{-\omega})^2\Omega(q^{-\omega})} \asymp -\frac{q^d\Omega(1)\Gamma(c)}{L^2\rho P'(\rho)\Omega(\rho)(\omega-c)^2} - \frac{\alpha q^d\Omega(1)\Gamma(c)}{L\rho P'(\rho)\Omega(\rho)(\omega-c)} - \frac{q^d\Omega(1)\Gamma'(c)}{L^2\rho P'(\rho)\Omega(\rho)(\omega-c)}.$$

This gives the following contribution to the asymptotic expansion of $\tilde{h}(z)$:

$$-\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}(\log_Q z)z^{-\log_Q \rho}\sum_k \Gamma(\log_Q \rho + \chi_k)z^{-\chi_k} + \frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}z^{-\log_Q\rho}\sum_k\left(\alpha\Gamma(\log_Q\rho + \chi_k) + \frac{\Gamma'(\log_Q\rho+\chi_k)}{L}\right)z^{-\chi_k}.$$

The second term has simple poles at c with singularity expansion

$$\Gamma(\omega)\left(-\sum_{l\geq 1}\frac{\Omega(q^{-\omega+l})q^{-\omega+l}P'(q^{-\omega+l})\bar{F}(\omega-l)}{P(q^{-\omega})\Omega(q^{-\omega})} - \frac{q^d\Omega(1)\alpha_p}{P(q^{-\omega})\Omega(q^{-\omega})}\right) \asymp \frac{q^d\Omega(1)\Gamma(c)}{L\rho P'(\rho)\Omega(\rho)(\omega-c)}(\alpha - \alpha_p).$$

(In the corresponding computation in [12], there occurred a small mistake, since $\alpha - \alpha_p$ was taken to be zero. The present version corrects this.) Hence, the contribution of this term to the asymptotic expansion of $\tilde{h}(z)$ is

$$-\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}(\alpha-\alpha_p)z^{-\log_Q\rho}\sum_k\Gamma(\log_Q\rho+\chi_k)z^{-\chi_k}.$$

Adding up the two contributions gives

$$\tilde{h}(z) \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}(\log_Q z)z^{-\log_Q\rho}\sum_k\Gamma(\log_Q\rho+\chi_k)z^{-\chi_k} + \frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}z^{-\log_Q\rho}\sum_k\left(\alpha_p\Gamma(\log_Q\rho+\chi_k) + \frac{\Gamma'(\log_Q\rho+\chi_k)}{L}\right)z^{-\chi_k}.$$

Next, note that due to Lemma 2 and the closure properties of Lemma 2.3 in [6], $\tilde{h}(z)$ is JS-admissible. Consequently, by Proposition 2.2 in [6],

$$\sum_k kp_{n,k} \sim -\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}(\log_Q n)n^{-\log_Q \rho}\sum_k \Gamma(\log_Q \rho + \chi_k)n^{-\chi_k}$$
$$+\frac{q^d\Omega(1)}{L\rho P'(\rho)\Omega(\rho)}n^{-\log_Q \rho}\sum_k \left(\alpha_p\Gamma(\log_Q \rho + \chi_k) + \frac{\Gamma'(\log_Q \rho + \chi_k)}{L}\right)n^{-\chi_k}.$$

Dividing by p_n and using Theorem 1 yields

$$\frac{\sum_k kp_{n,k}}{p_n} \sim \log_Q n - \alpha_p - \frac{1}{L}\frac{\sum_k \Gamma'(\log_Q \rho + \chi_k)n^{-\chi_k}}{\sum_k \Gamma(\log_Q \rho + \chi_k)n^{-\chi_k}}.$$

Finally, by pulling out the average value in the last term on the right-hand side, we obtain our second main result.

Theorem 2. *The expected value of the largest letter in a random word of length n satisfying the restricted growth property is asymptotically given by*

$$\mathbb{E}(X_n) \sim \log_Q n - \alpha_p - \frac{\psi(\log_Q \rho)}{L} + \Phi(\log_Q n),$$

where $\Phi(z)$ is a 1-periodic function with average value equal to zero and $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the Gamma function.

Remark 2. Again $\Phi(z)$ is a periodic function with very small amplitude.

4. Conclusion

Let us go back to

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^{d} pq^{l-1}\tilde{f}(q^l z).$$

We set

$$\tilde{f}(z) = \sum_{n\ge 0} b_n \frac{z^n}{n!};$$

then

$$b_{n+1} + b_n = \sum_{l=1}^{d} pq^{l-1}q^{ln}b_n,$$

and

$$b_n = -\Big(1 - \frac{p}{q}\sum_{l=1}^{d} q^{ln}\Big)b_{n-1} = (-1)^n\prod_{j=1}^{n}\Big(1 - \frac{p}{q}\sum_{l=1}^{d} q^{lj}\Big).$$

So an explicit expression *is* available even here. However, it would be unpleasant to work with it, especially when considering the parameter maximal letter, whereas the approach, as discussed in this paper, is still quite manageable.

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