PROTECTION NUMBER IN PLANE TREES

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Abstract. The protection number of a plane tree is the minimal distance of the root to a leaf; this definition carries over to an arbitrary node in a plane tree by considering the maximal subtree having this node as a root. We study the the protection number of a uniformly chosen random tree of size n and also the protection number of a uniformly chosen node in a uniformly chosen random tree of size n. The method is to apply singularity analysis to appropriate generating functions. Additional results are provided as well.

1. Introduction

Cheon and Shapiro [2] started the study of 2-protected nodes in trees. A node enjoys this property if its distance to any leaf is at least 2. After this pioneering paper, a large number of papers has been published [13, 6, 1, 10, 5, 12].

In this paper we study the protection number of the root of a (rooted, plane) tree (in the older literature often called ordered tree): It is the minimal distance of the root to any leaf. Further, the protection number of any node is defined by taking the (maximal) subtree that has this node as the root.

Preliminary results on the subject have appeared in the recent paper [3], but we show that, thanks to a rigorous use of methods outlined in the book *Analytic Combinatorics* [9], we can go much further. We are able to solve a basic recurrence explicitly, which allows us to use singularity analysis of generating functions and getting, at least in principle, as many terms as one wants in the asymptotic expansions of interest. Further, one can derive explicit expressions for the probabilities in question.

Some curious observations related to the constants that appear are also made; they are linked to identities due to Dedekind, Ramanujan and others and are part of the toolkit of the *modern analysis of algorithms*.

2. Results

In a rooted plane tree t, a vertex is said to be k-protected if its minimum distance from a leaf is at least k. The tree t is said to be k-protected if its root is k-protected.

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We denote the maximal k such that a vertex v is k-protected by $\pi(v)$ and call it the protection number of v. The protection number of the root is called the protection number of the tree, $\pi(t)$. This means that a tree t is k-protected if and only if $\pi(t) \geq k$.

If a vertex v is a leaf, then $\pi(v) = 0$ by definition. Otherwise, if it has children $w_1, \ldots, w_\ell$, then

$$\pi(v) = 1 + \min\{\pi(w_1), \ldots, \pi(w_\ell)\}.$$

We are interested in the random variables

- X_n, the protection number of a uniformly chosen random tree with n vertices,
- Y_n, the protection number of a uniformly chosen vertex in a uniformly chosen random tree with n vertices.

We will prove the following results.

Theorem 1. *For $n \to \infty$, the protection number X_n of a tree with n vertices tends to a discrete limit distribution:*

$$\begin{aligned}\mathbb{P}(X_n = k) = {} & \frac{27 \cdot 4^k (4^{2k} - 1)}{(4^k + 2)^2 (2 \cdot 4^k + 1)^2} \\ & + \tfrac{81 \cdot 4^k (4(k-3)4^{6k} + 36 \cdot 4^{5k} - (45k-72)4^{4k} - 80k4^{3k} - (45k+72)4^{2k} - 36 \cdot 4^k + 4(k+3))}{2(4^k+2)^4 (2 \cdot 4^k + 1)^4} \frac{1}{n} \\ & + O\Big(\frac{k^2}{3^k n^{3/2}}\Big).\end{aligned}$$

Setting

$$\begin{aligned}c_0 &= \sum_{k \geq 1} \frac{9 \cdot 4^k}{(4^k + 2)^2} \\ &= 1.6229713847153530495146582031843459896355136689840635394078 25\ldots, \\ c_1 &= \sum_{k \geq 1} \frac{9 \cdot 4^k ((3k-8)4^{2k} + 28 \cdot 4^k - (12k + 20))}{2(4^k + 2)^4} \\ &= 0.1311873689494231825244485810366733833577429413531428274982796\ldots, \\ c_2 &= \sum_{k \geq 1} \frac{9(2k-1)4^k}{(4^k + 2)^2} - c_0^2 \\ &= 0.7156950717833326673154891986827362860106611878542261743107 5\ldots, \\ c_3 &= \sum_{k \geq 1} \frac{9(2k-1)4^k ((3k-8)4^{2k} + 28 \cdot 4^k - (12k + 20))}{2(4^k + 2)^4} - 2c_0 c_1 \\ &= -0.2946393227325953234338781857554581438294988551586440707052 18\ldots,\end{aligned}$$

its expectation and variance can be written as

$$\begin{aligned}\mathbb{E}(X_n) &= c_0 + c_1\frac{1}{n} + O\Big(\frac{1}{n^{3/2}}\Big),\\ \mathbb{V}(X_n) &= c_2 + c_3\frac{1}{n} + O\Big(\frac{1}{n^{3/2}}\Big).\end{aligned}$$

Theorem 2. *For $n \to \infty$, the protection number Y_n of a random vertex of a random tree with n vertices tends to a discrete limit distribution:*

$$\begin{aligned}\mathbb{P}(Y_n = k) = {} & \frac{9 \cdot 4^k}{(4 \cdot 4^k + 2)(4^k + 2)}\\ & + \frac{3 \cdot 4^k(4^k - 1)((6k - 22)4^{3k} + (21k + 30)4^{2k} + (21k + 96)4^k + (6k + 58))}{(4^k + 2)^3(2 \cdot 4^k + 1)^3}\frac{1}{n}\\ & + O\Big(\frac{k^2}{3^k n^2}\Big).\end{aligned}$$

Setting

$$\begin{aligned}d_0 &= \sum_{k\ge 1}\frac{3}{4^k + 2}\\ &= 0.7276492769137260975311844004821453488635157227750422765370 08\ldots,\\ d_1 &= \sum_{k\ge 1}\frac{(3k - 10)4^{2k} + (6k + 26)4^k - 16}{2(4^k + 2)^3}\\ &= -0.031183712598622277494524648993610043742589912871352172530717 5\ldots,\\ d_2 &= \sum_{k\ge 1}\frac{3(2k - 1)}{4^k + 2} - d_0^2\\ &= 0.8168993794836289227887920562332298353956262869103163164075 7\ldots,\\ d_3 &= \sum_{k\ge 1}\frac{(2k - 1)((3k - 10)4^{2k} + (6k + 26)4^k - 16)}{2(4^k + 2)^3} - 2d_0d_1\\ &= 0.014197899249123624176745586362758197533680269252844749278840\ldots,\end{aligned}$$

its expectation and variance can be written as

$$\begin{aligned}\mathbb{E}(Y_n) &= d_0 + d_1\frac{1}{n} + O\Big(\frac{1}{n^{3/2}}\Big),\\ \mathbb{V}(Y_n) &= d_2 + d_3\frac{1}{n} + O\Big(\frac{1}{n^{3/2}}\Big).\end{aligned}$$

3. Number of k-Protected Trees

In this section, we investigate the auxiliary quantity r_{nk}, the number of k-protected trees with n vertices.

Let $\mathcal{R}_{\geq k}$ denote the set of all rooted plane trees with protection number $\geq k$. Its generating function is denoted by $R_{\geq k}(z)$ where z labels the number of vertices of a tree, i.e.,

$$R_{\geq k}(z) = \sum_{n\geq 1} r_{nk} z^n.$$

Lemma 3.1. *We have*

$$R_{\geq 0}(z) = \frac{1-\sqrt{1-4z}}{2}, \tag{1}$$

and

$$R_{\geq k}(z) = \frac{(1-z)z^{k-2}(R_{\geq 0}(z))^3}{1+z^{k-2}(R_{\geq 0}(z))^3} \tag{2}$$

for $k \geq 1$.

Proof. It is clear that $R_{\geq 0}$ is the generating function of all rooted plane trees which is well-known to be given by (1), cf. for instance [9, § I.5.1].

For $k \geq 1$, the root of a tree is k-protected if and only if all of its children are $(k-1)$-protected. Thus a tree is k-protected if and only if it consists of a root and a non-empty sequence of branches whose roots are $(k-1)$-protected. This translates into the symbolic equation shown in Figure 1 and thus into the equation

$$R_{\geq k}(z) = \frac{zR_{\geq k-1}(z)}{1-R_{\geq k-1}(z)} \tag{3}$$

for $k \geq 1$.

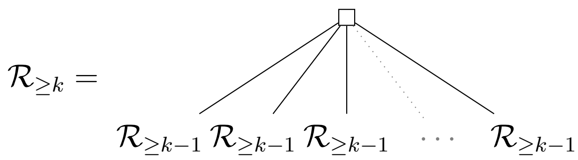

FIGURE 1. Symbolic equation for $\mathcal{R}_{\geq k}$.

It would now be easy to prove (2) by induction; however, we intend to *derive* (2).

We rewrite the recurrence (3) in the form

$$R_{\geq k}(z) = -z + \frac{z}{1-R_{\geq k-1}(z)}$$

such that $R_{\geq k-1}$ occurs only once.

Using the abbreviation $F_k := R_{\geq k}(z) + z$ this leads to

$$F_k = \frac{z}{1+z-F_{k-1}}.$$

This is reminiscent of continued fractions. We use the Ansatz $F_k = za_k/a_{k+1}$ resulting in the equation

$$\frac{za_k}{a_{k+1}} = \frac{za_k}{(1+z)a_k - za_{k-1}}.$$

It is sufficient to require

$$a_{k+1} = (1+z)a_k - za_{k-1}$$

for $k \geq 1$.

This is a linear recurrence whose characteristic equation has roots 1 and z, so its solution has the form $a_k = A + Bz^k$. As common factors between a_k and a_{k+1} do not matter, we may choose $A = z^3$ which leads to $B = (R_{\geq 0}(z))^3$.

Thus

$$R_{\geq k}(z) = \frac{z(z^3 + z^k(R_{\geq 0}(z))^3)}{z^3 + z^{k+1}(R_{\geq 0}(z))^3} - z$$

which results in (2). □

Proposition 3.2. *The probability that a tree is k protected is*

$$\begin{aligned}\mathbb{P}(X_n \geq k) = {}&\frac{9\cdot 4^k}{(4^k+2)^2}\\ &+\frac{9\cdot 4^k((3k-8)4^{2k}+28\cdot 4^k-(12k+20))}{2(4^k+2)^4}\frac{1}{n}\\ &+O\Big(\frac{k^2}{3^k n^{3/2}}\Big).\end{aligned}\tag{4}$$

Proof. We intend to use singularity analysis ([8], [9, Chapter VI]). Let z be in some Δ-domain at 1/4 (see [9, Definition VI.1]) with $|z - 1/4| \leq 1/12$. We have

$$\begin{aligned}z^{k-2} &= \Big(\frac{1}{4} - \Big(\frac{1}{4} - z\Big)\Big)^{k-2}\\ &= \frac{16}{4^k} - \frac{16(k-2)}{4^k}(1-4z) + O\Big(\frac{k^2}{3^k}(1-4z)^2\Big).\end{aligned}$$

Inserting this into (2), we get

$$\begin{aligned}R_{\geq k}(z) = {}&\frac{3}{2\cdot 4^k+4} + \frac{-9\cdot 4^k}{2\cdot 4^{2k}+8\cdot 4^k+8}(1-4z)^{1/2}\\ &+\frac{-3k\cdot 4^{2k}-6k4^k+16\cdot 4^{2k}-20\cdot 4^k+4}{2\cdot 4^{3k}+12\cdot 4^{2k}+24\cdot 4^k+16}(1-4z)\\ &+\frac{9k4^{3k}-24\cdot 4^{3k}-36k4^k+84\cdot 4^{2k}-60\cdot 4^k}{2\cdot 4^{4k}+16\cdot 4^{3k}+48\cdot 4^{2k}+64\cdot 4^k+32}(1-4z)^{3/2}\\ &+O\Big(\frac{k^2}{3^k}(1-4z)^2\Big).\end{aligned}\tag{5}$$

By singularity analysis, we obtain

$$r_{nk} = \Bigl(\frac{9\cdot 4^k}{4\sqrt{\pi}(4^{2k}+4\cdot 4^k+4)}\Bigr)4^n n^{-\frac{3}{2}} + \Bigl(\frac{9(12\cdot 4^{3k}k-29\cdot 4^{3k}+124\cdot 4^{2k}-48\cdot 4^k k-68\cdot 4^k)}{32\sqrt{\pi}(4^{4k}+8\cdot 4^{3k}+24\cdot 4^{2k}+32\cdot 4^k+16)}\Bigr)4^n n^{-\frac{5}{2}} + O\Bigl(\frac{k^2 4^n}{3^k n^3}\Bigr)$$

Singularity analysis and division by the number C_{n-1} (the $(n-1)$st Catalan-number) of rooted plane trees (this corresponds to setting $k=0$ above) yields (4).

□

Proof of Theorem 1. We use $\mathbb{P}(X_n = k) = \mathbb{P}(X_n \ge k) - \mathbb{P}(X_n \ge k+1)$ and Proposition 3.2 to prove the limit theorem.

The expectation follows from the well-known formula

$$\mathbb{E}(X_n) = \sum_{k\ge 1}\mathbb{P}(X_n \ge k)$$

which is valid for all random variables with non-negative integer values.

Similarly, the variance follows from $\mathbb{V}(X_n) = \mathbb{E}(X_n^2) - \mathbb{E}(X_n)^2$ and

$$\mathbb{E}(X_n^2) = \sum_{k\ge 1} k^2\mathbb{P}(X_n = k) = \sum_{k\ge 1}(2k-1)\mathbb{P}(X_n \ge k).$$

□

4. Protection Numbers of all Vertices

We now turn to the protection numbers of arbitrary vertices. We fix some k and consider the number s_{nk} of k-protected vertices summed over all trees of size n. The corresponding generating function is denoted by

$$S_{\ge k}(z) = \sum_{n\ge 0} s_{nk} z^n$$

where z again labels the number of vertices.

Lemma 4.1. *We have*

$$S_{\ge k}(z) = \frac{1}{2}R_{\ge k}(z)\Bigl(1+\frac{1}{\sqrt{1-4z}}\Bigr). \tag{6}$$

Proof. In the language of the symbolic method, the generating function $S_{\ge k}(z)$ corresponds to the class $\Theta_{\mathcal{Z}_{\ge k}}\mathcal{T}$ where $\Theta_{\ge\mathcal{Z}_k}$ denotes *pointing* to a k-protected vertex, cf. [9, Definition I.14].

A tree t and a k-protected vertex w in this tree bijectively correspond to a k-protected tree t_1 whose root is merged with a leaf of another tree t_2, cf. Figure 2. Thus $\mathcal{Z}\times\Theta_{\ge k}\mathcal{T}$ corresponds bijectively to $\mathcal{R}_{\ge k}\times\Theta_{\mathcal{V}}\mathcal{T}$ where $\Theta_{\mathcal{V}}$ denotes pointing at a leaf and the factor

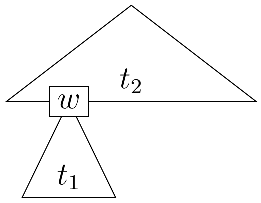

FIGURE 2. Decomposition at a k-protected vertex.

$\mathcal{Z}$ on the left hand side denotes one single vertex which compensates the fact that merging the root of one tree with a leaf of the other tree reduces the number of vertices by 1.

Thus $S_{\geq k}(z) = z^{-1}R_{\geq k}(z)L(z)$ where $L(z)$ denotes the generating function of $\Theta_{\mathcal{V}}\mathcal{T}$ with respect to the number of vertices. Note that the pointing is with respect to leaves, but L is a generating function with respect to all vertices.

Let

$$T(v, z) = \sum_{n\geq 0}\sum_{\ell\geq 0} N_{n-1,\ell}v^{\ell}z^{n}$$

denote the generating function of $\mathcal{T}$ where z marks the number of vertices and v marks the number of leaves. Here, $N_{n-1,\ell}$ denotes the Narayana number counting the number of trees with n vertices and ℓ leaves.

It is a well-known consequence of the symbolic method that

$$T(v, z) = zv + \frac{zT(v, z)}{1 - T(v, z)},$$

cf. [9, Example III.13]. This yields the explicit expression

$$T(v, z) = \frac{1 - z + vz - \sqrt{(v-1)^2z^2 - 2(v+1)z + 1}}{2}.$$

Pointing corresponds to applying $v\frac{d}{dv}$, cf. [9, Theorem I.4]. Setting $v = 1$ then leads to

$$L(z) = \Big(v\frac{dT(v, z)}{dv}\Big)\Big|_{v=1} = \frac{z}{2}\Big(1 + \frac{1}{\sqrt{1-4z}}\Big).$$

This yields (6). ☐

Proposition 4.2. *The probability that a random vertex of a random tree with n vertices has protection number at least k is*

$$\mathbb{P}(Y_n \geq k) = \frac{3}{4^k + 2} + \frac{(3k-10)4^{2k} + (6k+26)4^k - 16}{2(4^k+2)^3}\frac{1}{n} + O\Big(\frac{k^2}{3^kn^2}\Big).$$

Proof. By (6) and (5), we get

$$\begin{aligned}
S_{\geq k}(z) &= \frac{3}{4\cdot 4^k+8}(1-4z)^{1/2} - \frac{3\cdot 4^k-3}{2\cdot 4^{2k}+8\cdot 4^k+8} \\
&\quad - \frac{(3k-7)4^{2k}+(6k+38)4^k-4}{4\cdot 4^{3k}+24\cdot 4^{2k}+48\cdot 4^k+32}(1-4z)^{-1/2} \\
&\quad + \frac{(3k-4)4^{3k}-6(k-8)4^{2k}-24(k+2)4^k+4}{2\cdot 4^{4k}+16\cdot 4^{3k}+48\cdot 4^{2k}+64\cdot 4^k+32}(1-4z)^{-1} \\
&\quad + O\Big(\frac{k^2}{3^k(1-4z)^{-3/2}}\Big).
\end{aligned}$$

By singularity analysis, we get

$$\begin{aligned}
s_{nk} = \frac{3}{4\sqrt{\pi}(4^k+2)}4^n n^{-1/2} \\
&+ \frac{(12k-31)4^{2k}+(24k+140)4^k-28}{32\sqrt{\pi}(4^{3k}+6\cdot 4^{2k}+12\cdot 4^k+8)}4^n n^{-3/2} + O\Big(\frac{k^2}{3^k n^{5/2}}\Big).
\end{aligned}$$

Dividing by the number C_{n-1} of all trees and by the number n of vertices yields (4.2). ☐

Proof of Theorem 2. Theorem 2 follows from Proposition 4.2 in the same way as Theorem 1 follows from Proposition 3.2. ☐

5. Explicit formula for the number of $\geq k$-protected trees

Our goal is to read off the coefficient of z^n in formula (2) in explicit form.

Proposition 5.1. *The number of k-protected trees with n vertices is*

$$r_{nk} = \sum_{j\geq 1}(-1)^{j-1}\left[\binom{2n-3-(2k-1)j}{n-(k+1)j} - \binom{2n-3-(2k-1)j}{n-3-(k+1)j}\right].$$

Proof. We use the substitution $z = u/(1+u)^2$, which was introduced in [4] and rewrite (2) as

$$\begin{aligned}
G_{\geq k}\left(\frac{u}{(1+u)^2}\right) &= \frac{1-u^3}{(1-u)(1+u)^2}\frac{\frac{u^{k+1}}{(1+u)^{2k-1}}}{1+\frac{u^{k+1}}{(1+u)^{2k-1}}} \\
&= \frac{1-u^3}{(1-u)(1+u)^2}\sum_{j\geq 1}(-1)^{j-1}\frac{u^{(k+1)j}}{(1+u)^{(2k-1)j}}.
\end{aligned}$$

Extracting coefficients is now done with the Cauchy integral formula:

$$
\begin{aligned}
&[z^n]\frac{1-u^3}{(1-u)(1+u)^2}\sum_{j\ge1}(-1)^{j-1}\frac{u^{(k+1)j}}{(1+u)^{(2k-1)j}}\\
&=\frac{1}{2\pi i}\oint\frac{dz}{z^{n+1}}\frac{1-u^3}{(1-u)(1+u)^2}\sum_{j\ge1}(-1)^{j-1}\frac{u^{(k+1)j}}{(1+u)^{(2k-1)j}}\\
&=\frac{1}{2\pi i}\oint\frac{du}{u^{n+1}}(1+u)^{2n-3}(1-u^3)\sum_{j\ge1}(-1)^{j-1}\frac{u^{(k+1)j}}{(1+u)^{(2k-1)j}}\\
&=\sum_{j\ge1}(-1)^{j-1}[u^{n-(k+1)j}](1-u^3)(1+u)^{2n-3-(2k-1)j}.
\end{aligned}
$$

Using the binomial theorem yields the assertion. ☐

6. Functional equations for the constants

Two of the constants satisfy attractive and non-trivial functional equations. This phenomenon is not uncommon in the analysis of algorithms; we point out the paper [11] where it was first observed and the survey [14] which contains many references to earlier papers.

The first example is the constant

$$c_0=\frac{9}{2}\sum_{k\ge1}\frac{2^{2k-1}}{(2^{2k-1}+1)^2}=\frac{9}{2}F(\log 2),$$

with

$$F(x):=\sum_{k\ge1}\frac{e^{(2k-1)x}}{(e^{(2k-1)x}+1)^2}=\sum_{k\ge1}\frac{e^{-(2k-1)x}}{(1+e^{-(2k-1)x})^2}=\sum_{k,j\ge1}(-1)^{j-1}je^{-(2k-1)jx}.$$

Proposition 6.1. *We have the functional equation*

$$F(x)=\frac{1}{4x}-\frac{\pi^2}{x^2}F\Big(\frac{\pi^2}{x}\Big).$$

Since $\frac{\pi^2}{\log^2 2}F\big(\frac{\pi^2}{\log 2}\big)=0.0000134525077\ldots$, we have the *near-identity* $F(\log 2)\approx\frac{1}{4\log 2}$.

Proof. We compute the Mellin transform of it [7], which exists (at least) in the fundamental strip $\langle 2,\infty\rangle$:

$$F^*(s)=\Gamma(s)\sum_{k,j\ge1}(-1)^{j-1}j(2k-1)^{-s}j^{-s}=\Gamma(s)\zeta(s-1)\zeta(s)(1-2^{2-s})(1-2^{-s}).$$

The inversion formula for the Mellin transform gives the original function back (integration is along vertical lines). We shift the line of integration to the left and collect residues:

$$\begin{aligned} F(x) &= \frac{1}{2\pi i}\int_{(\frac{5}{2})} \Gamma(s)\zeta(s-1)\zeta(s)(1-2^{2-s})(1-2^{-s})x^{-s}ds \\ &= \frac{1}{4x} + \frac{1}{2\pi i}\int_{(-\frac{5}{2})} \Gamma(s)\zeta(s-1)\zeta(s)h(s)x^{-s}ds \end{aligned}$$

for

$$h(s) = (1-2^{2-s})(1-2^{-s}).$$

In the remainder of this proof, the relation

$$h(2-s) = 2^{2s-2}h(s) \tag{7}$$

will be the only property of $h(s)$ that we will use.

We now use the duplication formula for the Gamma function and the functional equation for the Riemann zeta function, a substitution $s = 2-u$ and then again a shift of the line of integration.

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(-\frac{5}{2})} \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)\zeta(s-1)\zeta(s)h(s)x^{-s}ds \\ &\qquad = \frac{1}{2\pi i}\int_{(-\frac{5}{2})} \frac{2^{s-1}}{\sqrt{\pi}}\frac{s-1}{2}h(s)\pi^{2s-2}\Gamma\Big(\frac{1-s}{2}\Big)\zeta(1-s)\Gamma\Big(\frac{2-s}{2}\Big)\zeta(2-s)x^{-s}ds \\ &\qquad = \frac{1}{2\pi i}\int_{(\frac{9}{2})} \frac{2^{1-u}}{\sqrt{\pi}}\frac{1-u}{2}h(2-u)\pi^{2-2u}\Gamma\Big(\frac{u-1}{2}\Big)\zeta(u-1)\Gamma\Big(\frac{u}{2}\Big)\zeta(u)x^{u-2}du \\ &\qquad = -\frac{1}{2\pi i}\int_{(\frac{5}{2})} 2^{u-1}h(u)\pi^{2-2u}\frac{1}{2^{u-1}}\Gamma(u)\zeta(u-1)\zeta(u)x^{u-2}du \\ &\qquad = -\frac{\pi^2}{x^2}\frac{1}{2\pi i}\int_{(\frac{5}{2})} h(u)\pi^{-2u}\Gamma(u)\zeta(u-1)\zeta(u)x^{u}du \\ &\qquad = -\frac{\pi^2}{x^2}F\Big(\frac{\pi^2}{x}\Big). \end{aligned}$$

□

Our second example relates to a sum that appears within the constant d_2:

$$S = \frac{3}{2}\sum_{k\ge1}\frac{2k-1}{2^{2k-1}+1} = \frac{3}{2}\sum_{k\ge1}\frac{(2k-1)2^{-2k+1}}{1+2^{-2k+1}} = \frac{3}{2}\sum_{k,j\ge1}(-1)^{j-1}(2k-1)2^{-j(2k-1)} = \frac{3}{2}G(\log 2)$$

with

$$G(x) = \sum_{k,j\geq 1} (-1)^{j-1}(2k-1)e^{-j(2k-1)x}.$$

Proposition 6.2. *We have the functional equation*

$$G(x) = \frac{\pi^2}{24x^2} + \frac{1}{24} - \frac{\pi^2}{x^2} G\Big(\frac{\pi^2}{x}\Big).$$

Since $\frac{\pi^2}{\log^2 2} G\big(\frac{\pi^2}{\log 2}\big) = 0.0000134525165276\ldots$, we have the near-identity $G(\log 2) \approx \frac{\pi^2}{24 \log^2 2} + \frac{1}{24}$.

Proof. The Mellin transform of G is

$$\Gamma(s) \sum_{k,j\geq 1} (-1)^{j-1} j^{-s} (2k-1)^{-s+1} = \Gamma(s)\zeta(s)\zeta(s-1)(1-2^{1-s})^2.$$

The proof of Proposition 6.1 applies with $h(s)$ replaced by $(1-2^{1-s})^2$ which again has the property (7). $\square$

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