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ORDERED FIBONACCI PARTITIONS

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ABSTRACT. Ordered partitions are enumerated by $F_n = \sum_k k!\, S(n,k)$ where $S(n,k)$ is the Stirling number of the second kind. We give some comments on several papers dealing with ordered partitions and turn then to ordered Fibonacci partitions of $\{1,\ldots,n\}$: If d is a fixed integer, the sets A appearing in the partition have to fulfill $i, j \in A,\ i \neq j \Rightarrow |i-j| \geq d$. The number of ordered Fibonacci partitions is determined.

1. The polynomials $F_n(x)=\sum_k k!\, S(n,k)x^k$ and the numbers $F_n := F_n(1)$ have appeared in the literature for several times ($S(n,k)$ are *Stirling numbers* of the second kind [2]): R. D. James [5] dealt with F_n considering the number of *ordered nontrivial factorizations of a squarefree integer.* In O. A. Gross [3] F_n appears as the total number of *distinct rational preferential arrangements*; this connection was recently rediscovered by J. P. Bartholemy [1]. The *Bell numbers* B_n [6] are counting the number of ways to partition the set $\{1,\ldots,n\}$. Now $S(n,k)$ is the number of partitions of $\{1,\ldots,n\}$ into exactly k blocks. Thus $B_n=\sum_k S(n,k)$. This shows a very close relationship between B_n and F_n. F_n counts each partition with k blocks with the factor $k!$ which refers to the number of ways to permute the blocks. So F_n can be interpreted as the total number of *ordered partitions of a set with n elements* (compare S. M. Tanny [9]).

In this note we first give some comments on the previous papers dealing with F_n and $F_n(x)$ and turn then to the case of *ordered d-Fibonacci partitions* of a set with n elements (cf. [7], [8]): We allow only those ordered paritions where the blocks $A \subseteq \{1,\ldots,n\}$ satisfy $i, j \in A,\ i \neq j \Rightarrow |i-j| \geq d$. Let $F_n^{(d)}$ be the number of those ordered partitions. Our main result is

$$F_n^{(d)} = 2^{1-d} \sum_{k=0}^{d-1} |s(d, d-k)|\, F_{n-k}$$

where $|s(d,l)|$ are signless *Stirling numbers* of the first kind.

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2. It was observed [1], [3], [5] that

$$G(z)=\sum_{n\geq 0} F_n \frac{z^n}{n!}=\frac{1}{2-e^z}. \tag{1}$$

From this the asymptotic behaviour of F_n was derived [1], [3], [5]:

$$F_n \sim \frac{n!}{2}\left(\frac{1}{\log 2}\right)^{n+1}, \quad (n\to\infty). \tag{2}$$

Using the method of *subtracted singularities* (Henrici [4]), a stronger result is most easily derived: Regarding the zeros of $2-e^z$, we find that $G(z)$ has singularities at $z_k=\log 2+2k\pi i$, $k\in\mathbb{Z}$. The singularities in question are just simple poles; the local expansions about those poles are

$$G(z)=\frac{1/2}{z_k-z}+O(1), \qquad (z\to z_k). \tag{3}$$

The knowledge of the local behaviour about the singularities gives enough information to grind out an asymptotic formula for F_n with an arbitrary small error term (by choosing $m\in\mathbb{N}$). We find

$$\frac{F_n}{n!}=\frac{1}{2}\sum_{|k|<m} z_k^{-(n+1)}+O(z_m^{-n}), \quad (n\to\infty). \tag{4}$$

S. M. Tanny [9] gives for $x\neq -1$ the following representation of $F_n(x)$ as an infinite series:

$$F_n(x)=\frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k x^n. \tag{5}$$

As pointed out in [9], this formula is only meaningful for $|x/(1+x)|<1$, i.e. $\operatorname{Re} x>-1/2$.

We give now a similar formula which is valid for $|(x+1)/x|<1$, i.e. $\operatorname{Re} x<-1/2$:

Let $A(n,k)$ be the *Eulerian numbers* ([2]) and $A_n(u):=\sum_k A(n,k)u^k$. A formula of *Frobenius* ([2]) *gives*

$$A_n(u)=u\sum_{k=1}^{n} k!\,S(n,k)(u-1)^{n-k}, \tag{6}$$

from which we conclude that

$$F_n(x)=\frac{x^{n+1}}{x+1}A_n\left(\frac{x+1}{x}\right). \tag{7}$$

Now it is well known that (e.g. see [2])

$$\frac{A_n(u)}{(1-u)^{n+1}}=\sum_{k\geq 0} u^k k^n, \tag{8}$$

which gives after simplification

$$F_n(x)=\frac{(-1)^{n+1}}{1+x}\sum_{k\geq 0}\left(\frac{1+x}{x}\right)^k k^n. \tag{9}$$

We remark that from (7) and the definition of $A_n(u)$ formula (16) of [9] is most easily derived.

We give yet another formula for F_n. For this, let $[z^n]f$ denote the coefficient of z^n in the power series f.

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{k}{i}&=[z^n]\frac{1}{1-z}\sum_{k\geq 0}(-1)^k\binom{k}{i}z^k\\&=[z^n]\frac{1}{1-z}\frac{1}{i!}(-z)^i\left(\frac{d}{d(-z)}\right)^i\frac{1}{1+z}\\&=[z^n]\frac{1}{1-z}\frac{1}{i!}(-z)^i\frac{i!}{(1+z)^{i+1}}\\&=[z^n]\frac{(-z)^i}{(1-z)(1+z)^{i+1}}.\end{aligned} \tag{10}$$

Now

$$k!\,S(n,k)=\sum_{i\geq 0}i^n(-1)^{k-i}\binom{k}{i} \tag{11}$$

and thus $(n\geq 1)$

$$\begin{aligned}F_n=\sum_{k=0}^{n}k!\,S(n,k)&=\sum_{i\geq 0}(-1)^i i^n\sum_{k=0}^{n}(-1)^k\binom{k}{i}\\&=\sum_{i\geq 0}i^n[z^n]\frac{z^i}{(1-z)(1+z)^{i+1}}\\&=[z^n]\frac{1}{(1-z)(1+z)}\sum_{i\geq 0}\left(\frac{z}{1+z}\right)^i i^n\\&=[z^n]\frac{1}{(1-z)(1+z)}A_n\left(\frac{z}{1+z}\right)\cdot(1+z)^{n+1}\\&=[z^n]\frac{(1+z)^n}{1-z}A_n\left(\frac{z}{1+z}\right).\end{aligned} \tag{12}$$

3. In [7], [8] the present writer defined a *d-Fibonacci set* $A\subseteq\{1,2,\dots,n\}$ to be a set with the property

$$i,j\in A,\qquad i\neq j\Rightarrow |i-j|\geq d. \tag{13}$$

The numbers $C_n^{(d)}$ of partitions of $\{1,\dots,n\}$ where all sets are d-Fibonacci sets were determined; it turned out that

$$C_n^{(d)}=B_{n+1-d}, \tag{14}$$

where B_m is a *Bell number*. Within this context, it is natural to consider $F_n^{(d)}$, the number of *ordered partitions of* $\{1, \ldots, n\}$ *into* d*-Fibonacci sets*. These numbers are most easily determined by use of a particularly elegant technique developed by Rota [6].

Let us recall from [7] that the number of functions $f:\{1, \ldots, n\} \to U$ (a finite set with u elements) such that

$$|\{f(i), f(i+1), \ldots, f(i+d-1)\}| = d \text{ for all } i \tag{15}$$

is given by

$$(u)_{d-1}(u-d+1)^{n+1-d}, \tag{16}$$

where $(u)_d := u(u-1)\cdots(u-d+1)$.

The functions fulfilling (15) are partitioned with respect to their kernels: (The kernel of f is the partition of $\{1, \ldots, n\}$ defined by saying that a and b are in the same block iff $f(a)=f(b)$.) Let $N(\pi)$ denote the number of blocks of the partition π. Then

$$\sum (u)_{N(\pi)} = (u)_{d-1}(u-d+1)^{n+1-d}. \tag{17}$$

The application of the linear functional $\bar{L}$ defined by $(u)_k \to 1$ for all k to (17) gives $C_n^{(d)}$, since each summand gives a contribution of 1. To find $F_n^{(d)}$, we have to use the linear functional L defined by $(u)_k \to k!$, because there are $k!$ ways to "order" the k blocks of the partition, so that the contribution of a partition with $N(\pi)$ blocks to the application of L to the left-handside of (17) is $N(\pi)!$.

Tanny [9] has proved that for any polynomial p

$$L(p(u)) = p(0) + L(\Delta p(u)) \tag{18}$$

with $\Delta p(u) = p(u+1) - p(u)$. Repeated application of (18) gives:

Let s be the smallest natural number such that $p(s) \neq 0$ holds (for $p \not\equiv 0$); then

$$2^s Lp(u) = Lp(u+s). \tag{19}$$

Now we have

$$F_n^{(d)} = L(u)_{d-1}(u-d+1)^{n+1-d} \tag{20}$$

and thus using (19) with $p(u) = (u)_{d-1}$ and $s = d-1$

$$2^{d-1} F_n^{(d)} = L(u+d-1)_{d-1} u^{n+1-d}. \tag{21}$$

In Comtet [2] we find essentially that

$$(u+d-1)_{d-1} = \sum_{k=0}^{d-1} |s(d, k+1)|\, u^k, \tag{22}$$

where the $|s(d, l)|$ are *signless Stirling numbers of the first kind*. From this we infer

$$F_n^{(d)} = 2^{1-d} L \sum_{k=0}^{d-1} |s(d, k+1)| \, u^{n+1-d+k}$$

(23)
$$= 2^{1-d} L \sum_{k=0}^{d-1} |s(d, d-k)| \, u^{n-k}$$

$$= 2^{1-d} \sum_{k=0}^{d-1} |s(d, d-k)| \, F_{n-k}.$$

An easy consequence of (2) and (23) is

(24)
$$F_n^{(d)} \sim 2^{1-d} F_n, \quad (n \to \infty).$$

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