A BIJECTION BETWEEN TWO SUBFAMILIES OF MOTZKIN PATHS

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Abstract. Two subfamilies of Motzkin paths, with the same numbers of up, down, horizontal steps were known to be equinumerous with ternary trees and related objects. We construct a bijection between these two families that does not use any auxiliary objects, like ternary trees.

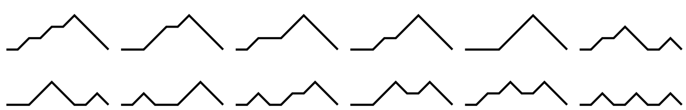
1. Introduction

Motzkin paths are similar to Dyck paths, but allow also horizontal steps of unit length. In this note, we concentrate on two subfamilies, where there are n up steps (u), n down steps (d), and n horizontal steps (h). Both families are enumerated by

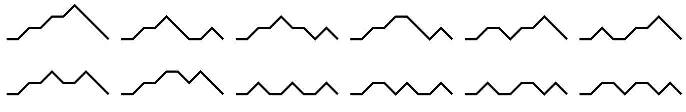
$$1 + z^3 + 3z^6 + 12z^9 + 55z^{12} + 273z^{15} + \cdots ,$$

and the coefficients also enumerate ternary trees and many other objects, see sequence A001764 in [2].

The first family originates from Asinowski and Mansour [1]. They start from a Dyck path of length $2n$ and *label* each maximal sequence of up steps by a Dyck path. If we say *replace* instead of *label* and use the steps h and u for the replaced sequence, we have a Motzkin path with n horizontal steps. To clarify, we give a list of all 12 such paths of length 9:



The second family was introduced to model *frog hops* from a question in a student's olympiad [3]: They were called S-Motzkin paths, have the same number of u, d, h, and when deleting the down steps, the sequence must look like $huhuhu\dots hu$. Here is again a list of all 12 objects of length 9:



These paths and some of their properties were investigated in the recent paper [4].

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The goal of the present note is to describe a bijection between the two families, which operates strictly on the paths themselves, without involving any other objects that are equinumerous.

Instead of drawing pictures, we use the more economical description with the letters u, d, h, and we always consider the paths from left to right.

2. From paths of the Asinowski/Mansour type to S-Motzkin paths

From paths given by Asinowski and Mansour, let $A_0D_1A_1D_2\ldots A_{t-1}D_t$ denote a path where A_i $(i=0,1,\ldots,t-1)$ denotes a sub-path consisting of horizontal and up steps, and D_i $(i=1,2,\ldots,t)$ denotes a sub-path consisting of consecutive down steps. Let d_i $(i=1,2,\ldots,t)$ denote the number of steps of D_i. For completeness, we mention that all A_i and D_i are non-empty.

First, we replace horizontal steps and up steps by up steps and down steps, respectively, in A_i to get a Dyck path $\overline{A_i}$. Then for each $\overline{A_i}$, we insert horizontal steps into it to form an S-Motzkin path A_i' such that each horizontal step is in the first position that can be inserted from left to right. Observe that in A_i', there are no two horizontal steps on the same height such that all the steps between these two steps are above that height, and each up step except for the last one must be followed (eventually) by a horizontal step.

These canonically obtained S-Motzkin paths are the *building stones* of the final object, and the numbers $d_1, d_2, \ldots, d_{t-1}$ are used to tell us how to *glue* them together. Notice also that the number d_t will not be used for the construction.

Let u_i denote the number of up steps of A_i'. Notice that

$$\sum_{k=1}^{i} d_k \le \sum_{k=0}^{i-1} u_k \quad (1 \le i \le t-1), \tag{2.1}$$

$$\sum_{k=1}^{t} d_k = \sum_{k=0}^{t-1} u_k. \tag{2.2}$$

First, we draw A_0'. Then according to $d_1, d_2, \ldots, d_{t-1}$, we insert $A_1, A_2, \ldots, A_t$ in turn. If $d_1 < u_0$, then from the beginning of A_0', find the d_1-th up step and insert A_1' behind it. If $d_1 = u_0$, then insert A_1' behind the last step of A_0'.

Assume that we have inserted A_{i-1} $(1 \le i \le t-1)$. Then from the beginning of A_{i-1}', find the d_i-th up step. Notice that if $d_i < u_{i-1}$, then the d_i-th up step belongs to A_{i-1}' and is not the last up step of A_{i-1}'. We insert A_i' behind this d_i-th up step. If $d_i = u_{i-1}$, then insert A_i' behind the end of A_{i-1}'. If $d_i > u_{i-1}$, then the d_i-th up step may belong to A_0', A_1', …, or A_{i-2}'. This time, if this step is not any last up step of A_0', A_1', …, or A_{i-2}', we insert A_i' behind it. Otherwise, if this step is the last up step of A_s' $(s \in \{0,1,\ldots,i-2\})$, then we put A_i' behind A_s'. Note that (2.1) and (2.2) ensure that we can always find the aimed up step.

Finally, we get an S-Motzkin path until we have inserted all A_i' $(i=0,1,\ldots,t-1)$.

3. From S-Motzkin paths to paths of the Asinowski/Mansour type

For an S-Motzkin path P, from left to right, we check the horizontal steps in turn. For a given horizontal step h_i, look along the path from h_i. If there is a closest horizontal step on the same height and all the steps between these two steps denoted by P_i are not below this height, then h_iP_i is an S-Motzkin path. We call h_i a paired horizontal step. For the first horizontal step of the S-Motzkin path P, no matter whether it is paired or not, we always call it a paired horizontal step denoted by h_0. If there are no other horizontal steps on height 0, then $P = h_0P_0$. Let $h_0, h_1, \ldots, h_{t-1}$ denote these paired horizontal steps of P.

If there are no paired horizontal steps in P_i, then h_iP_i is an S-Motzkin sub-path with all the horizontal steps located in their first positions. We assume h_iP_i as A'_i. Back to the S-Motzkin path P, if the up step just before h_i is the d_i-th up step when we are counting from the beginning of A'_{i-1}, then we record d_i. Then in the S-Motzkin path P, we find the next paired horizontal step behind h_i, and repeat the process.

If there is a paired horizontal step h_{i+1} in P_i, then we use h_{i+1} to find A'_{i+1}. Note that now A'_i contains the steps of h_iP_i without those steps in A'_{i+1}.

In view of all the paired horizontal steps, and using the above method, we can identify all A'_i $(i = 0, 1, \ldots, t-1)$ and d_i $(i = 1, 2, \ldots, t-1)$.

Deleting all the horizontal steps in A'_i $(0 \leq i \leq t-1)$, and then replacing up and down steps by horizontal and up steps, respectively, we obtain A_i. Let D_i $(1 \leq i \leq t-1)$ be the sub-path consisting of d_i consecutive down steps. Then we draw a path $A_0D_1A_1D_2A_2 \ldots D_{t-1}A_{t-1}$, and add enough consecutive down steps after it to form a Motzkin path given by Asinowski and Mansour.

4. A detailed example

We consider a path given by Asinowski and Mansour

$$hhhuuu\mathbf{d}hhuuhu\mathbf{d}hu\mathbf{dd}hhuu\mathbf{ddddd}.$$

Then we can divide the paths into four parts A_0, A_1, A_2 and A_3 by using the consecutive down steps which are in bold. Set

$$\begin{array}{llll} A_0 = hhhuuu, & A_1 = hhuuhu, & A_2 = hu, & A_3 = hhuu, \\ d_1 = 1, & d_2 = 1, & d_3 = 2, & d_4 = 5. \end{array}$$

In fact, we do not need d_4 in the bijection.

First, replacing horizontal steps and up steps by up steps and down steps, respectively, in A_i $(i = 0, 1, 2, 3)$, we have four Dyck paths

$$\overline{A_0} = uuuddd, \quad \overline{A_1} = uuddud, \quad \overline{A_2} = ud, \quad \overline{A_3} = uudd.$$

Then inserting horizontal steps in $\overline{A}_i$ $(i = 0, 1, 2, 3)$ in the first possible positions, we get

$$A'_0 = huhuhuddd, \quad A'_1 = huhuhddud, \quad A'_2 = hud, \quad A'_3 = huhudd.$$

Since $d_1 = 1$, we find the first up step in A'_0. Then inserting A'_1 behind this step, we obtain

$$hu\mathbf{huhuhddud}huhuddd.$$

For the above path, since $d_2 = 1$, we find the first up step from the beginning of A'_1. That is to say, deleting the first two steps in the above path, we get a sub-path. For this sub-path, we find the d_2-th up step, and then insert A'_2. Let u_1 be the number of up steps of A'_1. Note that if $d_2 < u_1$, then A'_2 is inserted in A'_1. If $d_2 = u_1$, then put A'_2 just behind A'_1. If $d_2 > u_1$, then A_2 is inserted between two steps of A'_0. Back to this example, we have $d_2 < u_1$. So

$$huhu\mathbf{hud}huhddudhuhuddd.$$

For the above path, since $d_3 = 2$, we find the second up step from the beginning of A'_2. That is to say, deleting the first four steps in the above path yields a sub-path. In this sub-path, we find the second up step. In this example, $d_3 > u_2$, and the second up step is not the last up step of A'_1 or A'_0. So we insert A'_3 behind the second up step directly. we have

$$huhuhudhu\mathbf{huhudd}hddudhuhuddd$$

which is an S-Motzkin path.

Inversely, for the S-Motzkin path

$$huhuhudhuhuhuddhddudhuhuddd,$$

we can find the paired horizontal steps as follows:

$$\mathbf{h}u\mathbf{h}u\mathbf{h}udhu\mathbf{h}uhuddhddudhuhuddd.$$

The first horizontal step must be a paired horizontal step although there is no other horizontal step on level 0. We mark it as

$$(\bar{h}uhuhudhuhuhuddhddudhuhuddd).$$

Here we use a pair of parentheses to identify A'_0.

Then we find the next pair of horizontal steps

$$(\bar{h}u(\check{h}uhudhuhuhuddhddud)\check{h}uhuddd).$$

So we have $d_1 = 1$, and we use a pair of parentheses to identify A'_1 from A'_0.

Next, we have

$$(\bar{h}u(\check{h}u(\hat{h}ud)\hat{h}uhuhuddhddud)\check{h}uhuddd)$$

and $d_2 = 1$. We add a pair of parentheses to identify A'_2.

Finally, we obtain

$$(\bar{h}u(\check{h}u(\hat{h}ud)\hat{h}u(\dot{h}uhudd)\dot{h}ddud)\check{h}uhuddd)$$

and $d_3 = 2$ where we use a pair of parentheses to identify A'_3.

Therefore, we have

$A'_0 = huhuhuddd,$	$A'_1 = huhuhddud,$	$A'_2 = hud,$	$A'_3 = huhudd,$
$d_1 = 1,$	$d_2 = 1,$	$d_3 = 2.$	

Deleting all the horizontal steps in A'_0, A'_1, A'_2 and A'_3, and then replacing up and down steps by horizontal and up steps, respectively, we obtain

$$A_0 = hhhuuu, \qquad A_1 = hhuuhu, \qquad A_2 = hu, \qquad A_3 = hhuu.$$

Combining A_i $(i = 0, 1, 2, 3)$ and d_i $(i = 1, 2, 3)$, we have the following path

$$hhhuuu\mathbf{d}hhuuhu\mathbf{d}hu\mathbf{dd}hhuu.$$

Finally, we add enough down steps at the end of the above path to form a Motzkin path given by Asinowski and Mansour:

$$hhhuuu\mathbf{d}hhuuhu\mathbf{d}hu\mathbf{dd}hhuu\mathbf{ddddd}.$$

5. The objects of length 9 matched

For the reader's convenience, we provide the correspondence of 12 objects. The number 12 is very convenient; it is not too small and not too large. In [4], there were also many explicit lists with 12 objects each.

No.	AM-paths	AM-paths dec.	S-Motzkin paths
1	*huhuhuddd*	*ududud* 3	*huhduhdud*
2	*hhuuhuddd*	*uuddud* 3	*huhuhddud*
3	*huhhuuddd*	*uduudd* 3	*huhduhudd*
4	*hhuhuuddd*	*uududd* 3	*huhuhdudd*
5	*hhhuuuddd*	*uuuddd* 3	*huhuhuddd*
6	*huhuddhud*	*udud* 2 *ud* 1	*huhdudhud*
7	*hhuuddhud*	*uudd* 2 *ud* 1	*huhuddhud*
8	*hudhhuudd*	*ud* 1 *uudd* 2	*hudhuhudd*
9	*hudhuhudd*	*ud* 1 *udud* 2	*hudhuhdud*
10	*hhuudhudd*	*uudd* 1 *ud* 2	*huhudhudd*
11	*huhudhudd*	*udud* 1 *ud* 2	*huhudhdud*
12	*hudhudhud*	*ud* 1 *ud* 1 *ud* 1	*hudhudhud*

Table 1: Paths from Asinowski and Mansour, also decomposed, and the corrresponding S-Motzkin paths.

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