DEPTH AND PATH LENGTH OF m-PLANE ORIENTED RECURSIVE TREES

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Abstract. An m-version of plane oriented recursive trees is considered. Moments of depth and path length are computed asymptotically. Two transformations on the level of generating functions from the ordinary ($m = 1$) case are used.

1. Introduction

Cichon and Macyna [3], in the context of *approximate counting*, introduced a parameter m, which means that m counters are used instead of just one. The association of an incoming element to one of the counters is done at random, with probability $\frac{1}{m}$. This brilliant idea is of course not restricted to approximate counting. Recently, the second author undertook a study of binary search trees, where m such trees are kept, and again incoming (new) elements are inserted into a random tree.

Here, we continue this line of research, by considering *Plane Oriented Recursive Trees*, shortly PORTs. These trees also appeared under the name *heap ordered trees*, [2, 9], but we agree with Hwang [8] that PORT is a much more appropriate name. They belong to the family of *increasing trees* [1].

A PORT is created from the list $1, 2, \ldots, n$. Assume that a PORT with $n - 1$ elements has already been created. The new element n can be attached to any node; if such a node has d successors, it can be put in one of the $d + 1$ slots between them. It follows that there are

$$a_n = 1 \cdot 3 \cdot 5 \cdots (2n - 3) = n! 2^{1-n} C_n$$

such trees, with a Catalan number

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Following the general paradigm, we now construct m PORTs; a new element is randomly assigned to one of the m PORTs, and each PORT is created as usual.

We analyse two parameters: the depth (distance from the root) of a random element, and the path length. The path length of an ordinary PORT is the sum of the distances of the nodes to the root, and for the m-version, it is the sum of the path lengths of the individual PORTs. We follow the paper [9]; it turns out that the results derived there

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can be used. They do not need to be rederived. One must control carefully how the two counting problems change when switching to m PORTs. That is what we will do in this paper.

In the above mentioned paper [10], explicit formulæ could be derived for the m-BST version. Here, that is not possible. We must resort to asymptotic methods, and we do this via generating functions and singularity analysis of them [5]. There is a dominant singularity at 1, and known expansions of the typical terms. We have to transform generating functions from [9] twice; first, they have to be adapted to become *ordinary generating functions*, and then the m-version follows from something that is often called *Euler transform* [6].

2. Known results about PORTs

We cite from the paper [9]: The expectation of the depth of a random node in a PORT of size n is given by

$$\mathbb{E}(D_n) = \Big(1-\frac{1}{2n}\Big)\widehat{H}_n - \frac{1}{2};$$

the variance is

$$\mathbb{V}(D_n) = \Big(1-\frac{1}{2n}\Big)\Big[\widehat{H}_n - \widehat{H}^{(2)}_n\Big] - \mathbb{E}(D_n)^2,$$

with

$$\widehat{H}_n = \sum_{k=1}^{n}\frac{1}{2k-1} \quad\text{and}\quad \widehat{H}^{(2)}_n = \sum_{k=1}^{n}\frac{1}{(2k-1)^2}.$$

These quantities can be expressed in terms of traditional harmonic numbers,

$$H_n = \sum_{k=1}^{n}\frac{1}{k} \quad\text{and}\quad H^{(2)}_n = \sum_{k=1}^{n}\frac{1}{k^2},$$

but it is useful to have a special notation here.

The expectation of the path length of a PORT of size n is given by

$$\mathbb{E}(P_n) = \Big(n-\frac{1}{2}\Big)\widehat{H}_n - \frac{n}{2};$$

the variance is

$$\mathbb{V}(P_n) = n^2\Big(\frac{3}{2} - \widehat{H}^{(2)}_n\Big) + n\Big(\widehat{H}^{(2)}_n - \widehat{H}_n - \frac{3}{4}\Big) + \frac{1}{2}\widehat{H}_n - \frac{1}{4}\widehat{H}^{(2)}_n.$$

We cite here the known expansions involving $\widehat{H}_n$ and $\widehat{H}^{(2)}_n$ since we will need them later. Our reference is [7].

$$[z^n]\frac{1}{(1-z)^{\alpha+1}}\log\frac{1}{1-z} = \binom{n+\alpha}{n}(H_{n+\alpha}-H_\alpha),$$

$$[z^n]\frac{1}{(1-z)^{\alpha+1}}\log^2\frac{1}{1-z} = \binom{n+\alpha}{n}\Big((H_{n+\alpha}-H_\alpha)^2 - (H^{(2)}_{n+\alpha}-H^{(2)}_\alpha)\Big).$$

We will need this for $\alpha = 0, 1, 2$, but also for $\alpha = -\frac{1}{2}, \frac{1}{2}$, when it takes this form:

$$[z^n]\frac{1}{(1-4z)^{1/2}}\log\frac{1}{1-4z} = \binom{2n}{n}2\widehat{H}_n,$$

$$[z^n]\frac{1}{(1-4z)^{3/2}}\log\frac{1}{1-4z} = \binom{2n}{n}(2n+1)2(\widehat{H}_{n+1}-1),$$

and

$$[z^n]\frac{1}{(1-4z)^{1/2}}\log^2\frac{1}{1-4z} = \binom{2n}{n}4(\widehat{H}_n^2 - \widehat{H}_n^{(2)}),$$

$$[z^n]\frac{1}{(1-4z)^{3/2}}\log^2\frac{1}{1-4z} = \binom{2n}{n}(2n+1)4\big((\widehat{H}_{n+1}-1)^2 - (\widehat{H}_{n+1}^{(2)}-1)\big).$$

These expansions were already discussed and used in the earlier paper [9].

3. Depth in m-PORTs

The derivation of the depth of a random node in a PORT is via *level polynomials* $L_n(u)$: the coefficient of u^k is the probability that there are k nodes on level k. They are given by

$$L_n(u) = \frac{1}{2(1+u)C_n}(-4)^n\binom{-u/2}{n} + \frac{1}{1+u},$$

with Catalan numbers

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}.$$

Note that $L_n(1) = n$, as it should.

For the analogous quantity in m-PORTs we compute

$$\begin{aligned}\mathcal{L}_N(u) &= \sum_{n_1+\cdots+n_m=N} m^{-N}\binom{N}{n_1,\ldots,n_m}\big(L_{n_1}(u)+\cdots+L_{n_m}(u)\big)\\ &= \sum_{n=0}^{N} m^{1-N}(m-1)^{N-n}\binom{N}{n}L_n(u),\end{aligned}$$

and $\mathcal{L}_N(u)/N$ is the probability generating function of interest, i. e., moments of the depth can be obtained from it by differentiation. It does not seem to be promising to try for a closed form expression of $\mathcal{L}_N(u)$ directly, but we can use this relation to compute moments, as we will now demonstrate.

Assume that the (ordinary) generating function

$$F(z) := \sum_{n\ge1} L_n'(1)z^n$$

is known. Then

$$\mathcal{L}_N'(1) = \sum_{n=0}^{N} m^{1-N}(m-1)^{N-n}\binom{N}{n}L_n'(1)$$

$$= \sum_{n=0}^{N} m^{1-N}(m-1)^{N-n}\binom{N}{n}[z^n]F(z)$$
$$= [z^N]\frac{m^2}{m-(m-1)z}F\Big(\frac{z}{m-(m-1)z}\Big),$$

as is easy to check. This is often called an Euler transform [6].

We have an explicit formula for $F(z)$. The corresponding formula

$$\frac{m^2}{m-(m-1)z}F\Big(\frac{z}{m-(m-1)z}\Big)$$

is not suitable anymore for explicit coefficients, but asymptotics can be easily derived, by singularity analysis. The dominant singularity $z=1$ is again, after this transformation, at $z=1$, and we get an expansion that can be used for asymptotics. In principle, as many terms as one wants for an asymptotic expansion could be derived, although with some effort. The same is true for the second factorial moments:

$$G(z) := \sum_{n\ge1} L''_n(1)z^n,$$

then we need to consider

$$\frac{m^2}{m-(m-1)z}G\Big(\frac{z}{m-(m-1)z}\Big).$$

To be more specific,

$$F(z) = \sum_{n\ge0}\Big[\Big(n-\frac{1}{2}\Big)\widehat{H}_n - \frac{n}{2}\Big]z^n = \frac{z^{3/2}}{2(1-z)^2}\log\frac{1}{1-z} + \frac{z^{3/2}}{(1-z)^2}\log(1+\sqrt{z}\,).$$

Then

$$\frac{m^2}{m-(m-1)z}F\Big(\frac{z}{m-(m-1)z}\Big) = \frac{1}{2(1-z)^2}\log\frac{1}{1-z} + \frac{2\log2-\log m}{2(1-z)^2} + O\Big(\frac{1}{1-z}\log\frac{1}{1-z}\Big).$$

For the coefficient of z^N this means

$$\frac{N+1}{2}(H_{N+1}-1) + \frac{2\log2-\log m}{2}(N+1) + O(\log N),$$

and we get an asymptotic formula for the average of the depth of a random node in an m-PORT by dividing this by N and using well known asymptotics for harmonic numbers.

The explicit formula for $F(z)$ from the given coefficients can be computed using Maple. We will follow this approach also for the second factorial moment where it is more complicated to get the explicit formula for $G(z)$. It was done using Maple, in particular the package GFUN [11], and some human interaction. At a first glance, it looks like a guessed result, but it can be made fully rigorous. We do not, however, want to spend too much space on that. After all, once such an explicit formula is known, by backwards engineering, the coefficients may be computed!

As demonstrated in the paper [9], all results come out from terms like

$$\frac{1}{(1-4z)^{\alpha+1}}\log^{\beta}\frac{1}{1-4z}, \tag{1}$$

for which we know explicit formulæ for the coefficients, which must then be divided by C_n. For computations related to expectation and variance, only $\beta = 0, 1, 2$ is needed, but higher moments would require arbitrary values. For instance, computing the second factorial moment, we encounter

$$\begin{aligned}
[z^n]M_2(z) &= [z^n]\frac{z}{2\sqrt{1-4z}} - [z^n]\frac{1}{8\sqrt{1-4z}}\log\frac{1}{1-4z} + [z^n]\frac{1}{16\sqrt{1-4z}}\log^2\frac{1}{1-4z}\\
&= [z^{n-1}]\frac{1}{2\sqrt{1-4z}} - [z^n]\frac{1}{8\sqrt{1-4z}}\log\frac{1}{1-4z} + [z^n]\frac{1}{16\sqrt{1-4z}}\log^2\frac{1}{1-4z}\\
&= \frac{1}{2}\binom{2n-2}{n-1} - \frac{1}{4}\binom{2n}{n}\widehat{H}_n + \frac{1}{16}\binom{2n}{n}[4\widehat{H}_n^2 - 4\widehat{H}_n^{(2)}]
\end{aligned}$$

and

$$\frac{1}{C_n}[z^n]M_2(z) = \frac{n}{2} - \frac{2n-1}{2}\widehat{H}_n + \frac{2n-1}{2}[\widehat{H}_n^2 - \widehat{H}_n^{(2)}].$$

The most complicated ingredient is then given by

$$\begin{aligned}
\sum_{n\ge 0}\frac{2n-1}{2}[\widehat{H}_n^2 - \widehat{H}_n^{(2)}]z^n &= \frac{z^{3/2}}{4(1-z)^2}\log^2\frac{1}{1-z} + \frac{z^{3/2}(1+2\log(2))}{2(1-z)^2}\log\frac{1}{1-z}\\
&\quad + \frac{z^{3/2}}{(1-z)^2}\mathrm{Li}_2\Big(\frac{1-\sqrt{z}}{2}\Big)\\
&\quad - \frac{z^{3/2}}{2(1-z)^2}\log^2(1+\sqrt{z}\,) + \frac{z^{3/2}(1+\log(2))}{(1-z)^2}\log(1+\sqrt{z}\,)\\
&\quad - \frac{z^{3/2}\pi^2}{12(1-z)^2} + \frac{z^{3/2}\log^2(2)}{2(1-z)^2}.
\end{aligned}$$

Here,

$$\mathrm{Li}_2(z) := \sum_{n\ge 1}\frac{z^n}{n^2}.$$

We provided a full catalogue of functions as in (1) that are needed. The second table provides then the expansion of the translated formula $f(z/(m-(m-1)z)$ around $z = 1$. From these, everything can be put together, and we only provide the most interesting results, since intermediate steps are often long and performed with Maple anyway.

We apply this to

$$M_2(z) = -\frac{\sqrt{1-4z}}{8} + \frac{1}{8\sqrt{1-4z}} - \frac{1}{8\sqrt{1-4z}}\log\frac{1}{1-4z} + \frac{1}{16\sqrt{1-4z}}\log^2\frac{1}{1-4z};$$

the transformed version is (leading terms only), after simplification

$$\begin{aligned}&-\frac{2\log(2)-\log(m)}{2(1-z)^2}\\ &+\frac{1}{4(1-z)^2}\log^2\frac{1}{1-z}+\frac{2\log(2)-\log(m)}{2(1-z)^2}\log\frac{1}{1-z}\\ &+\frac{(\log(m)-2\log(2))(\log(m)-2\log(2)-2)}{4(1-z)^2}-\frac{\pi^2}{12(1-z)^2}.\end{aligned}$$

The coefficient of z^N, and its asymptotics, divided by N is a long expression that we don't display, but the variance simplifies. Let us thus summarize the results about the first two moments of the depth of a random node:

Theorem 1. *Expectation and variance of the depth of a random node in m-PORTs of size N admit the following asymptotic expansions:*

$$\begin{aligned}\mathbb{E}(\mathcal{D}_N) &= \frac{1}{2}\log\frac{N}{m} + \frac{1}{2}\gamma + \log 2 - \frac{1}{2} + O\Big(\frac{\log N}{N}\Big),\\ \mathbb{V}(\mathcal{D}_N) &= \frac{1}{2}\log\frac{N}{m} + \frac{\gamma}{2} - \frac{\pi^2}{8} - \frac{1}{4} + \log 2 + O\Big(\frac{\log N}{N}\Big).\end{aligned}$$

4. Path length in m-PORTs

We start from the probability generating function $P_n(u)$ for the path length.

$$\mathcal{P}_N(u) = m^{1-N}\sum_{n=0}^{N}(m-1)^{N-n}\binom{N}{n}P_n(u),$$

Note that now $\mathcal{P}_N(1) = m$, so it needs to be normalized.

$P_n(u)$ is not really available, but we can compute moments.

The expectation is just n times the expectation of the depth, so we find the depth in the m-model on average:

$$\sim \frac{N}{2m}\Big(\log\frac{N}{m} + \gamma + 2\log 2 - 1\Big).$$

The second factorial moment is much more demanding. Its generating function (the Catalan version) is given in [9]. Here we only repeat the leading terms

$$\frac{1}{32}\frac{1}{(1-4z)^{3/2}}\log^2\frac{1}{1-z} + \frac{1}{16}\frac{1}{(1-4z)^{3/2}}\log\frac{1}{1-z} + \frac{3}{32}\frac{1}{(1-4z)^{3/2}};$$

for a full list of terms see the original paper.

$f(z)$	$a_n := [z^n]f(z)/C_n$	$\sum_{n\geq 1} a_n z^n$
$\sqrt{1-4z}$	-2	$\frac{-2z}{1-z}$
$\frac{1}{\sqrt{1-4z}}$	$2(2n-1)$	$\frac{2z(1+z)}{(1-z)^2}$
$\frac{1}{(1-4z)^{3/2}}$	$2(2n+1)(2n-1)$	$\frac{2z(3+6z-z^2)}{(1-z)^3}$
$\frac{1}{(1-4z)^{1/2}}\log\frac{1}{1-4z}$	$4(2n-1)\widehat{H}_n$	$\frac{4z^{3/2}}{(1-z)^2}\log\frac{1}{1-z} + \frac{8z^{3/2}}{(1-z)^2}\log(1+\sqrt{z}) + \frac{4z}{(1-z)^2}$
$\frac{1}{(1-4z)^{1/2}}\log^2\frac{1}{1-4z}$	$8(2n-1)(\widehat{H}_n^2 - \widehat{H}_n^{(2)})$	$\frac{4z^{3/2}}{(1-z)^2}\log^2\frac{1}{1-z} + \frac{8(1+2\log(2))z^{3/2}}{(1-z)^2}\log\frac{1}{1-z} + \frac{16z^{3/2}}{(1-z)^2}\mathrm{Li}_2\Big(\frac{1-\sqrt{z}}{2}\Big) - \frac{8z^{3/2}}{(1-z)^2}\log^2(1+\sqrt{z}) + \frac{16(1+\log(2))z^{3/2}}{(1-z)^2}\log(1+\sqrt{z}) + \frac{8\log^2(2)z^{3/2}}{(1-z)^2} - \frac{4\pi^2 z^{3/2}}{3(1-z)^2}$
$\frac{1}{(1-4z)^{3/2}}\log\frac{1}{1-4z}$	$4(2n+1)(2n-1) \times(\widehat{H}_{n+1}-1)$	$\frac{16z^{3/2}}{(1-z)^3}\log\frac{1}{1-z} + \frac{32z^{3/2}\log(1+\sqrt{z})}{(1-z)^3} - \frac{12z^2}{(1-z)^3} + \frac{4z}{(1-z)^3}$
$\frac{1}{(1-4z)^{3/2}}\log^2\frac{1}{1-4z}$	$8(2n-1)(2n+1) \times[(\widehat{H}_{n+1}-1)^2 - (\widehat{H}_{n+1}^{(2)}-1)]$	$\frac{16z^{3/2}}{(1-z)^3}\log^2\frac{1}{1-z} - \frac{16(1-4\log(2))z^{3/2}}{(1-z)^3}\log\frac{1}{1-z} + \frac{64z^{3/2}}{(1-z)^3}\mathrm{Li}_2\Big(\frac{1-\sqrt{z}}{2}\Big) - \frac{32z^{3/2}}{(1-z)^3}\log^2(1+\sqrt{z}) - \frac{32(1-2\log(2))z^{3/2}}{(1-z)^3}\log(1+\sqrt{z}) - \frac{16\pi^2 z^{3/2}}{3(1-z)^3} + \frac{32\log^2(2)z^{3/2}}{(1-z)^3} + \frac{48z^2}{(1-z)^3}$

$F(z)$	$F\big(\frac{z}{m-(m-1)z}\big)$ leading terms only
$\dfrac{-2z}{1-z}$	$-\dfrac{2}{m}\dfrac{1}{1-z}$
$\dfrac{2z(1+z)}{(1-z)^2}$	$\dfrac{4}{m^2}\dfrac{1}{(1-z)^2}$
$\dfrac{2z(3+6z-z^2)}{(1-z)^3}$	$\dfrac{16}{m^3}\dfrac{1}{(1-z)^3}$
$\dfrac{4z^{3/2}}{(1-z)^2}\log\dfrac{1}{1-z} + \dfrac{8z^{3/2}}{(1-z)^2}\log(1+\sqrt{z}\,) + \dfrac{4z}{(1-z)^2}$	$\dfrac{4}{m^2}\dfrac{1}{(1-z)^2}\log\dfrac{1}{1-z}$ $\dfrac{4(1+2\log(2)-\log(m))}{m^2}\dfrac{1}{(1-z)^2}$
$\dfrac{4z^{3/2}}{(1-z)^2}\log^2\dfrac{1}{1-z} + \dfrac{8(1+2\log(2))z^{3/2}}{(1-z)^2}\log\dfrac{1}{1-z} + \dfrac{16z^{3/2}}{(1-z)^2}\mathrm{Li}_2\Big(\dfrac{1-\sqrt{z}}{2}\Big) + \dfrac{8(1+2\log(2))z^{3/2}}{(1-z)^2}\log\dfrac{1}{1-z} - \dfrac{8z^{3/2}}{(1-z)^2}\log^2(1+\sqrt{z}\,) + \dfrac{16(1+\log(2))z^{3/2}}{(1-z)^2}\log(1+\sqrt{z}\,) + \dfrac{8\log^2(2)z^{3/2}}{(1-z)^2} - \dfrac{4\pi^2z^{3/2}}{3(1-z)^2}$	$\dfrac{4}{m^2}\dfrac{1}{(1-z)^2}\log^2\dfrac{1}{1-z} + \dfrac{8(1+2\log(2)-\log(m))}{m^2}\dfrac{1}{(1-z)^2}\log\dfrac{1}{1-z} + \dfrac{4(\log(m)-2\log(2))(\log(m)-2\log(2)-2)}{m^2(1-z)^2} - \dfrac{4\pi^2}{3m^2(1-z)^2}$
$\dfrac{16z^{3/2}}{(1-z)^3}\log\dfrac{1}{1-z} + \dfrac{32z^{3/2}\log(1+\sqrt{z}\,)}{(1-z)^3} - \dfrac{12z^2}{(1-z)^3} + \dfrac{4z}{(1-z)^3}$	$\dfrac{16}{m^3(1-z)^3}\log\dfrac{1}{1-z} - \dfrac{8(1-4\log(2)+2\log(m))}{m^3(1-z)^3}$
$\dfrac{16z^{3/2}}{(1-z)^3}\log^2\dfrac{1}{1-z} - \dfrac{16(1-4\log(2))z^{3/2}}{(1-z)^3}\log\dfrac{1}{1-z} + \dfrac{64z^{3/2}}{(1-z)^3}\mathrm{Li}_2\Big(\dfrac{1-\sqrt{z}}{2}\Big) - \dfrac{32z^{3/2}}{(1-z)^3}\log^2(1+\sqrt{z}\,) - \dfrac{32(1-2\log(2))z^{3/2}}{(1-z)^3}\log(1+\sqrt{z}\,) - \dfrac{16\pi^2z^{3/2}}{3(1-z)^3} + \dfrac{32\log^2(2)z^{3/2}}{(1-z)^3} + \dfrac{48z^2}{(1-z)^3}$	$\dfrac{16}{m^3(1-z)^3}\log^2\dfrac{1}{1-z} - \dfrac{16(1-4\log(2)+2\log(m))}{m^3(1-z)^3}\log\dfrac{1}{1-z} - \dfrac{16\pi^2}{3m^3(1-z)^3} + \dfrac{48}{m^3(1-z)^3} + \dfrac{16(2\log(2)-\log(m))(2\log(2)-\log(m)-1)}{m^3(1-z)^3}$

According to our prepared lists, we can translate that into ordinary generating functions:

$$\frac{1}{m}\frac{1}{(1-z)^3}\log\frac{1}{1-z}-\frac{1-4\log(2)+2\log(m)}{2m(1-z)^3}$$
$$+\frac{1}{m}\frac{1}{2(1-z)^3}\log^2\frac{1}{1-z}-\frac{1-4\log(2)+2\log(m)}{2m}\frac{1}{(1-z)^3}\log\frac{1}{1-z}$$
$$-\frac{\pi^2}{6m(1-z)^3}+\frac{3}{m(1-z)^3}+\frac{(2\log(2)-\log(m))(2\log(2)-\log(m)+1)}{2m(1-z)^3}.$$

From this we can read off the coefficients of z^N and evaluate it asymptotically. We don't show intermediate steps since it is mostly done with Maple, and eventually there is a lot of simplification.

Theorem 2. *Expectation and variance of the path length in m-PORTs of size N admit the following asymptotic expansions:*

$$\mathbb{E}(\mathcal{P}_N)=\frac{N}{2m}\Big(\log\frac{N}{m}+\gamma+2\log 2-1\Big)+O(\log N),$$
$$\mathbb{V}(\mathcal{P}_N)=\frac{N^2}{m^2}\Big(\frac{3}{2}-\frac{\pi^2}{8}\Big)+O(N\log N).$$

So, when concentrating only on the leading terms, m-PORTs behave in a very predictable way, namely n is just replaced by N/m. Such a simple dependency is not likely to persist, however, when making the effort to compute lower order terms. This seems to be a good student's project.

5. Conclusion

This was just the beginning of the analysis of m-PORTs. Many more things are waiting to be discovered.

The transition from

$$\sum_{n\ge 0}a_nz^n \quad \text{to} \quad \sum_{n\ge 1}\frac{a_n}{C_n}z^n$$

was done here via the coefficients. It would be interesting to do this on a purely analytic level. Most likely, techniques related to the Hadamard product as in [4] will play a role.

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