

THE LEVEL OF NODES IN INCREASING TREES REVISITED

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ABSTRACT. *Simply generated families of trees* are described by the equation $T(z) = \varphi(T(z))$ for their generating function. If a tree has n nodes, we say that it is increasing if each node has a label $\in \{1, \dots, n\}$, no label occurs twice, and whenever we proceed from the root to a leaf, the labels are increasing. This leads to the concept of *simple families of increasing trees*. Three such families are especially important: *recursive trees*, *heap ordered trees*, and *binary increasing trees*. They belong to the subclass of *very simple families of increasing trees*, which can be characterized in 3 different ways.

This paper contains results about these families as well as about polynomial families (the function $\varphi(u)$ is just a polynomial). The random variable of interest is the level of the node (labelled) j , in random trees of size $n \geq j$. For very simple families, this is independent of n , and the limiting distribution is Gaussian. For polynomial families, we can prove this as well for $j, n \rightarrow \infty$ such that $n-j$ is fixed. Additional results are also given. These results follow from the study of certain trivariate generating functions and Hwang's quasi power theorem. They unify and extend earlier results by Devroye, Mahmoud and others.

1. INTRODUCTION

Increasing trees are labelled trees where the nodes of a tree of size n are labelled by distinct integers of the set $\{1, \dots, n\}$ in such a way that each sequence of labels along any branch starting at the root is increasing. As the underlying tree model we use the so called *simply generated trees* (see [11]) but, additionally, the trees are equipped with increasing labellings. We will thus speak about *simple families of increasing trees*. A thorough study of families (= varieties) of increasing trees was conducted in [1].

Several important tree families, in particular *recursive trees*, *heap ordered trees* (also called plane-oriented recursive trees) and *binary increasing trees* (also called tournament trees) are special instances of simple families of increasing trees. We will specialize all our general results to these 3 families.

A survey of applications and results on recursive trees and heap ordered trees is given by Mahmoud and Smythe in [10]. These models are used, e. g., to describe the spread of epidemics, for pyramid schemes, and quite recently as a simplified growth model of the world wide web.

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In [2] resp. [9] the level of the largest node, i. e., the node with the label n , in a random recursive tree resp. a random heap ordered tree of size n was studied. In a rooted tree, the level of a node v , also called the depth of node v , is measured by the number of edges lying on the unique path from the root to node v . If we denote by D_n the random variable which measures the depth of node n in a random increasing tree of size n , then it was shown in [2, 9] for recursive trees resp. heap ordered trees that D_n satisfies a central limit law with mean and variance $\sim \log n$ resp. $\sim \frac{1}{2} \log n$. Since increasing trees can be considered as weighted trees, we always assume as the model of randomness that all increasing trees of size n are chosen proportionally to their weights.

In Section 4 we give a unified approach to the investigation of the level of the largest node for *general* increasing tree families. We obtain a simple formula for a suitable bivariate generating function of the probabilities $\mathbb{P}\{D_n = m\}$, which allows to reprove the above mentioned exact and limiting distribution results for D_n for the most interesting increasing tree families, but also gives easily (together with the quasi power theorem due to Hwang) a central limit law of this parameter for so called *polynomial families of increasing trees*, i. e., increasing trees, where the out-degree of every node is bounded by some bound d .

In Section 5 we extend our analysis and study the random variable $D_{n,j}$, which measures the level of node j in a random increasing tree of size $n \geq j$. Again we obtain for general increasing tree families a quite simple formula for a suitable trivariate generating function of the probabilities $\mathbb{P}\{D_{n,j} = m\}$. Once more it is an easy task to derive distribution results of $D_{n,j}$ for the most interesting tree families, where we obtain the well known fact that the distribution of $D_{n,j}$ is independent of n . Although it seems not out of reach to obtain more general results from this explicit form of the trivariate generating function (at least) for polynomial increasing tree families, we restrict ourselves in the present paper to show a Gaussian limiting distribution of $D_{n,j}$ for $n \rightarrow \infty$ and j the k -th largest node, i. e., $j := n + 1 - k$ with k fixed.

In Section 6 we pursue the question of finding increasing tree families, where the distribution of $D_{n,j}$ is *independent* of n . To do this we study increasing tree families \mathcal{T} which have the following property: if we cut off all nodes larger than node j in a random tree of size $n \geq j$ of \mathcal{T} we obtain a *random* tree of \mathcal{T} of size j .

It is well known that several increasing tree families, in particular recursive trees, heap ordered trees, and binary increasing trees, can be constructed via an *insertion process* or a *probabilistic growth rule*. Thus, more generally, we are interested to study increasing tree families with this property. This means that one considers families \mathcal{T} , which have the property that for every tree T' of size $n-1$ with vertices v_1, \dots, v_{n-1} there exist probabilities $p_{T'}(v_1), \dots, p_{T'}(v_{n-1})$, such that when starting with a random tree T' of size $n-1$, choosing a vertex v_i in T' according to the probabilities $p_{T'}(v_i)$ and attaching node n to it, we obtain a *random* increasing tree T of size n .

It turns out that these two properties of increasing tree families are equivalent and we characterize all such increasing tree families via the degree-weight generating functions. Since these degree-weight generating functions appeared already in [12] and [5], where the corresponding tree families were termed *very simple tree families*, we call such tree families now *very simple **increasing** tree families*. Furthermore, a third equivalent property of very simple increasing tree families was given in [5], which is repeated here in Lemma 5.

With our unified approach, we reprove several known results from the literature, to which we give explicit references. The remaining results are new, to the best of our knowledge.

We also want to mention that a related parameter, namely the depth of a *randomly chosen node* in a random increasing tree of size n was studied using a unified approach in [1], where limiting distribution results were obtained for polynomial families of increasing trees, recursive trees, and heap ordered trees.

2. PRELIMINARIES

Formally, a class \mathcal{T} of a simple family of increasing trees can be defined in the following way. A sequence of non-negative numbers $(\varphi_k)_{k \geq 0}$ with $\varphi_0 > 0$ is used to define the weight $w(T)$ of any ordered tree T by $w(T) = \prod_v \varphi_{d(v)}$, where v ranges over all vertices of T and $d(v)$ is the out-degree of v (we always assume that it exists a $k \geq 2$ with $\varphi_k > 0$). Furthermore, $\mathcal{L}(T)$ denotes the set of different increasing labellings of the tree T with distinct integers $\{1, 2, \dots, |T|\}$, where $|T|$ denotes the size of the tree T , and $L(T) := |\mathcal{L}(T)|$ its cardinality. Then the family \mathcal{T} consists of all trees T together with their weights $w(T)$ and the set of increasing labellings $\mathcal{L}(T)$.

For a given degree-weight sequence $(\varphi_k)_{k \geq 0}$ with a degree-weight generating function $\varphi(t) := \sum_{k \geq 0} \varphi_k t^k$, we define now the total weights by $T_n := \sum_{|T|=n} w(T) \cdot L(T)$. It follows then that the exponential generating function $T(z) := \sum_{n \geq 1} T_n \frac{z^n}{n!}$ satisfies the *autonomous* first order differential equation

$$T'(z) = \varphi(T(z)), \quad T(0) = 0. \quad (1)$$

Often it is advantageous to describe a simple family of increasing trees \mathcal{T} by the formal recursive equation

$$\mathcal{T} = \textcircled{1} \times \left(\varphi_0 \cdot \{\epsilon\} \dot{\cup} \varphi_1 \cdot \mathcal{T} \dot{\cup} \varphi_2 \cdot \mathcal{T} * \mathcal{T} \dot{\cup} \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \dot{\cup} \dots \right) = \textcircled{1} \times \varphi(\mathcal{T}), \quad (2)$$

where $\textcircled{1}$ denotes the node labelled by 1, \times the cartesian product, $*$ the partition product for labelled objects, and $\varphi(\mathcal{T})$ the substituted structure (see e. g., [14]).

By specializing the degree-weight generating function $\varphi(t)$ in (1) we get the basic enumerative results for the three most interesting increasing tree families:

- *Recursive trees* are the family of non-plane increasing trees such that all node degrees are

allowed. The degree-weight generating function is $\varphi(t) = \exp(t)$. Solving (15) gives

$$T(z) = \log\left(\frac{1}{1-z}\right), \quad \text{and} \quad T_n = (n-1)!, \quad \text{for } n \geq 1. \quad (3)$$

• *Heap ordered trees* are the family of plane increasing trees such that all node degrees are allowed. The degree-weight generating function is $\varphi(t) = \frac{1}{1-t}$. Equation (15) leads here to

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = \frac{(n-1)!}{2^{n-1}} \binom{2n-2}{n-1} = 1 \cdot 3 \cdot 5 \cdots (2n-3) = (2n-3)!!, \quad \text{for } n \geq 1. \quad (4)$$

• *Binary increasing trees* have the degree-weight generating function $\varphi(t) = (1+t)^2$. Thus it follows

$$T(z) = \frac{z}{1-z}, \quad \text{and} \quad T_n = n!, \quad \text{for } n \geq 1. \quad (5)$$

In order to give some results for the random variable $D_{n,j}$ studied in this paper for polynomial families of increasing trees we will require some of the results given in [1], which are collected in the sequel. First we remark that depending on the degree-weight generating function $\varphi(t)$, periodicity phenomena can occur. If $\varphi(t)$ is a function of t^q for some $q \geq 2$, such that $\varphi(t) = \psi(t^q)$ for some power series ψ , one says that $\varphi(t)$ is periodic and the maximum possible q is called the period (otherwise $\varphi(t)$ is called aperiodic, $q = 1$). For a period $q \geq 2$ one gets e. g., by applying the Lagrange inversion formula, that $T(z) = zT^*(z^q)$ for some power series T^* . Thus non-zero coefficients T_n can occur only if the congruence condition $n \equiv 1 \pmod{q}$ is satisfied. For the asymptotic behaviour of the coefficients T_n one can translate the behaviour of $T(z)$ in the neighbourhood of the dominant singularities (singularities with smallest modulus) via singularity analysis (see [3]).

The next theorem describes the location of the dominant singularities.

Theorem 1. [Bergeron et al.] *Given a polynomial degree-weight generating function $\varphi(t) = \sum_{0 \leq k \leq d} \varphi_k t^k$ with $\varphi_0 > 0$, $\varphi_d > 0$, $\varphi_k \geq 0$ for $0 < k < d$. The dominant real positive singularity of the function $T(z)$, solution of $T'(z) = \varphi(T(z))$, $T(0) = 0$, is then given by*

$$\rho = \int_0^\infty \frac{dt}{\varphi(t)}. \quad (6)$$

Furthermore, if $\varphi(t)$ is non periodic, ρ is the only dominant singularity of $T(z)$. If $\varphi(t)$ has period $q \geq 2$, then $T(z) = zT^(z^q)$, where $T^*(t)$ has a unique dominant singularity at $t = \rho^q$.*

From this theorem it follows that $T(z)$ is analytic in a domain larger than the disk of convergence if we slit at angles $2\pi m/q$: it exists a $\rho' > \rho$, such that $T(z)$ is analytic in the domain

$$D := \{z \in \mathbb{C} : |z| \leq \rho', z \neq r e^{i\phi} \text{ with } \rho \leq r \leq \rho', \phi = 2\pi m/q \text{ and } 0 \leq m < q\}.$$

The next theorem describes the behaviour of $T(z)$ near the dominant singularity ρ . For a polynomial degree-weight generating function $\varphi(t) = \sum_{0 \leq k \leq d} \varphi_k t^k$ of degree d we use throughout this paper the abbreviations

$$\delta := \frac{1}{d-1} \quad \text{and} \quad \eta := \left(\frac{\varphi_d \rho}{\delta} \right)^\delta. \quad (7)$$

Theorem 2. [Bergeron et al.] *Let $\varphi(t) = \varphi_0 + \dots + \varphi_d t^d$ with $\varphi_0 > 0$, $\varphi_d > 0$, $\varphi_k \geq 0$ for $0 < k < d$, be a polynomial degree-weight generating function with degree $d \geq 2$. Then in a complex neighbourhood of ρ , the solution $T(z)$ is of the form*

$$T(z) = \frac{1}{\Delta(z)} H(\Delta(z)), \quad \text{where} \quad \Delta(z) = \eta \left(1 - \frac{z}{\rho} \right)^\delta, \quad (8)$$

and $H(t) = \sum_{k \geq 0} h_k t^k$ is analytic at $t = 0$, with $h_0 = 1$.

Via singularity analysis, one gets then in the aperiodic case immediately the asymptotic behaviour of the coefficients T_n . If the degree generating functions $\varphi(t)$ has period $q \geq 2$ one has q dominant singularities, and their contributions have to be added. This leads to an extra factor q in the formula (9) as stated in the next theorem.

Theorem 3. [Bergeron et al.] *Let \mathcal{T} be a polynomial family associated with the degree-weight generating function $\varphi(t) = \varphi_0 + \dots + \varphi_d t^d$, $\varphi_0 > 0$, $\varphi_d > 0$, $\varphi_k \geq 0$ for $0 < k < d$, which has period $q \geq 2$. The quantities T_n of elements of \mathcal{T} with size n satisfy for $d \geq 2$ and $n \rightarrow \infty$ with $n \equiv 1 \pmod{q}$:*

$$\frac{T_n}{n!} = \frac{q}{\eta \Gamma(\delta)} \rho^{-n} n^{-1+\delta} \left(1 + \mathcal{O}(n^{-2\delta}) \right). \quad (9)$$

To obtain Gaussian limiting distribution results we use the so called quasi power theorem as proven in [4], which is stated below for the reader's convenience.

Theorem 4. [H. K. Hwang] *Let $\{X_n\}_{n \geq 1}$ be a sequence of integral random variables. Suppose that the moment generating function satisfies the asymptotic expression*

$$M_n(s) := \mathbb{E} (e^{X_n s}) = \sum_{m \geq 0} \mathbb{P}\{X_n = m\} e^{ms} = e^{H_n(s)} (1 + \mathcal{O}(\kappa_n^{-1})),$$

the \mathcal{O} -term being uniform for $|s| \leq \sigma$, $s \in \mathbb{C}$, $\sigma > 0$, where

- (i) $H_n(s) = U(s)\phi(n) + V(s)$, with $U(s)$ and $V(s)$ analytic for $|s| \leq \sigma$ and independent of n ; $U''(0) \neq 0$,
- (ii) $\phi(n) \rightarrow \infty$,
- (iii) $\kappa_n \rightarrow \infty$.

Under these assumptions, the distribution of X_n is asymptotically Gaussian with the given rate of convergence in the Kolmogorov metric:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{X_n - U'(0)\phi(n)}{\sqrt{U''(0)\phi(n)}} \leq x \right\} - \Phi(x) \right| = \mathcal{O} \left(\frac{1}{\kappa_n} + \frac{1}{\sqrt{\phi(n)}} \right).$$

Moreover, the mean and the variance of X_n satisfy

$$\mathbb{E}(X_n) = U'(0)\phi(n) + V'(0) + \mathcal{O}(\kappa_n^{-1}), \quad \mathbb{V}(X_n) = U''(0)\phi(n) + V''(0) + \mathcal{O}(\kappa_n^{-1}).$$

The distribution function of the standard normal distribution $\mathcal{N}(0,1)$ is here always denoted by $\Phi(x)$. Furthermore we use throughout this paper the abbreviations $x^{\overline{m}} := x(x-1)\cdots(x-m+1)$ resp. $x^{\underline{m}} := x(x+1)\cdots(x+m-1)$ for the falling resp. rising factorials, and $\begin{bmatrix} n \\ m \end{bmatrix}$ for the signless Stirling numbers of first kind.

3. RESULTS

We obtain the following characterization of very simple increasing tree families via the degree-weight generating function $\varphi(t)$:

Lemma 5. *The following three properties of a simple family of increasing trees \mathcal{T} are equivalent:*

- (1) *The total weights T_n of trees of size n of \mathcal{T} satisfy the equation*

$$\frac{T_{n+1}}{T_n} = c_1n + c_2,$$

with fixed constants c_1, c_2 , for all $n \in \mathbb{N}$.

- (2) *Starting with a random increasing tree T of size $n \geq j$ of \mathcal{T} and removing all nodes larger than j we obtain a random increasing tree T' of size j of \mathcal{T} .*
- (3) *The family \mathcal{T} can be constructed via an insertion process resp. a probabilistic growth rule.*

The family \mathcal{T} satisfies these (equivalent) properties and is thus a very simple family of increasing trees iff the degree-weight generating function $\varphi(t) = \sum_{k \geq 0} \varphi_k t^k$ is given by one of the following three formulæ, where c_1, c_2 are there the constants appearing in property (1).

Case A: $\varphi(t) = \varphi_0 e^{\frac{c_1 t}{\varphi_0}}$, for $\varphi_0 > 0, c_1 > 0, (\Rightarrow c_2 = 0)$,

Case B: $\varphi(t) = \varphi_0 \left(1 + \frac{c_2 t}{\varphi_0}\right)^d$, for $\varphi_0 > 0, c_2 > 0, d := \frac{c_1}{c_2} + 1 \in \{2, 3, 4, \dots\}$,

Case C: $\varphi(t) = \frac{\varphi_0}{\left(1 + \frac{c_2 t}{\varphi_0}\right)^{-\frac{c_1}{c_2} - 1}}$, for $\varphi_0 > 0, 0 < -c_2 < c_1$.

Recursive trees are ‘‘Case A,’’ for $\varphi_0 = 1, c_1 = 1$; binary increasing trees are ‘‘Case B,’’ for $\varphi_0 = 1, c_1 = 1, c_2 = 1, d = 2$, heap ordered trees are ‘‘Case C,’’ for $\varphi_0 = 1, c_1 = 2, c_2 = -1$.

Results for the level of node j in a random tree of size n of a very simple increasing tree family are given in the following theorem.

Theorem 6. *If \mathcal{T} is a very simple family of increasing trees with a degree-weight generating function given as in Lemma 5 then the distribution of $D_{n,j}$, which is the level of node j in*

a random tree of size $n \geq j$, is independent of n . The probability generating function $p(v) := \sum_{m \geq 0} \mathbb{P}\{D_{n,j} = m\}v^m$ is for $n \geq j$ given by

$$p(v) = \frac{(j - 2 + (\frac{c_2}{c_1} + 1)v)^{j-1}}{(j + \frac{c_2}{c_1} - 1)^{j-1}} = \prod_{i=1}^{j-1} \left(1 + \frac{(c_1 + c_2)(v - 1)}{c_1 i + c_2}\right),$$

and the probabilities $\mathbb{P}\{D_{n,j} = m\}$ are given by the following explicit formula:

$$\mathbb{P}\{D_{n,j} = m\} = \frac{(\frac{c_2}{c_1} + 1)^m}{(\frac{c_2}{c_1} + 1)^{j-1}} \begin{bmatrix} j - 1 \\ m \end{bmatrix}.$$

Moreover, $D_{n,j}$ is for $j \rightarrow \infty$ asymptotically Gaussian, where the rate of convergence is of order $\mathcal{O}(\frac{1}{\sqrt{\log j}})$:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{D_{n,j} - (\frac{c_2}{c_1} + 1) \log j}{\sqrt{(\frac{c_2}{c_1} + 1) \log j}} \leq x \right\} - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{\sqrt{\log j}}\right),$$

and the expectation $\mathbb{E}(D_{n,j})$ and the variance $\mathbb{V}(D_{n,j})$ satisfy

$$\mathbb{E}(D_{n,j}) = \left(\frac{c_2}{c_1} + 1\right) \log j + \mathcal{O}(1), \quad \mathbb{V}(D_{n,j}) = \left(\frac{c_2}{c_1} + 1\right) \log j + \mathcal{O}(1).$$

The next theorem gives results for the level of the k -th largest node in a random tree of size n of a polynomial family of increasing trees.

Theorem 7. *Given a polynomial family of increasing trees with degree-weight generating function $\varphi(t) = \varphi_0 + \dots + \varphi_d t^d$ with $d \geq 2$, $\varphi_0 > 0$, $\varphi_d > 0$ and $\varphi_i \geq 0$ for $0 < i < d$, which has period $q \geq 1$. The distribution of the random variable $D_{n,n+1-k}$, which is the level of the k -th largest node, i. e., the node labelled with $n + 1 - k$, in a random tree of size n , is for fixed $k \geq 1$ and $n \rightarrow \infty$ with $n \equiv 1 \pmod{q}$ asymptotically Gaussian, where the rate of convergence is of order $\mathcal{O}(\frac{1}{\sqrt{\log n}})$:*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{D_{n,n+1-k} - \frac{d}{d-1} \log n}{\sqrt{\frac{d}{d-1} \log n}} \leq x \right\} - \Phi(x) \right| = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right).$$

The expectation $\mathbb{E}(D_{n,n+1-k})$ and the variance $\mathbb{V}(D_{n,n+1-k})$ satisfy

$$\mathbb{E}(D_{n,n+1-k}) = \frac{d}{d-1} \log n + \mathcal{O}(1), \quad \mathbb{V}(D_{n,n+1-k}) = \frac{d}{d-1} \log n + \mathcal{O}(1).$$

The next sections contain the proofs of the above assertions.

4. THE LEVEL OF THE LARGEST NODE

4.1. Treating the recurrence. First we will obtain from the formal recursive description (2) of simple families of increasing trees a recurrence for the probabilities $\mathbb{P}\{D_n = m\}$, where D_n is the random variable that counts the level of node n in a random increasing tree of size n . For increasing trees of size n with root-degree r and subtrees with sizes

k_1, \dots, k_r , enumerated from left to right, where the largest node n lies in the leftmost subtree, we can reduce the computation of the probabilities $\mathbb{P}\{D_n = m\}$ to the probabilities $\mathbb{P}\{D_{k_1} = m - 1\}$: We get the total weight of the r subtrees and the root node $\varphi_r T_{k_1} \cdots T_{k_r}$ as a factor, multiplied by the number $\binom{n-2}{k_1-1, k_2, \dots, k_r}$ of order preserving relabellings of the r subtrees with labels $2, 3, \dots, n-1$ (the node n is reserved for the largest node in the first subtree and 1 for the root node) and divided by the total weight T_n of trees of size n . Since one has to consider also the cases where the largest node n is in the second, third, \dots , r -th subtree, we also get a factor r due to symmetry arguments. Summing up over all choices for k_1, \dots, k_r and the degree r of the root node gives the following recurrence (10):

$$\mathbb{P}\{D_n = m\} = \sum_{r \geq 1} r \varphi_r \sum_{\substack{k_1 + \dots + k_r = n-1, \\ k_1, \dots, k_r \geq 1}} \frac{T_{k_1} \cdots T_{k_r}}{T_n} \binom{n-2}{k_1-1, k_2, \dots, k_r} \mathbb{P}\{D_{k_1} = m-1\}, \quad (10)$$

for $n \geq 2$, with initial value $\mathbb{P}\{D_1 = m\} = \delta_{0,m}$.

We treat this recurrence via generating functions and introduce¹

$$N(z, v) := \sum_{n \geq 1} \sum_{m \geq 0} \mathbb{P}\{D_n = m\} T_n \frac{z^{n-1}}{(n-1)!} v^m. \quad (11)$$

Multiplying (10) with $T_n \frac{z^{n-2}}{(n-2)!} v^m$ and summing over $n \geq 2$ and $m \geq 0$ gives

$$\sum_{n \geq 2} \sum_{m \geq 0} \mathbb{P}\{D_n = m\} T_n \frac{z^{n-2}}{(n-2)!} v^m = \frac{\partial}{\partial z} N(z, v), \quad (12)$$

and

$$\begin{aligned} & \sum_{n \geq 2} \sum_{m \geq 0} \sum_{r \geq 1} r \varphi_r \sum_{\substack{k_1 + \dots + k_r = n-1, \\ k_1, \dots, k_r \geq 1}} \frac{T_{k_1} \cdots T_{k_r}}{T_n} \binom{n-2}{k_1-1, k_2, \dots, k_r} \times \\ & \quad \times \mathbb{P}\{D_{k_1} = m-1\} T_n \frac{z^{n-2}}{(n-2)!} v^m \\ &= v \sum_{r \geq 1} r \varphi_r \sum_{k_1 \geq 1} T_{k_1} \mathbb{P}\{D_{k_1} = m-1\} \frac{z^{k_1-1}}{(k_1-1)!} \sum_{k_2 \geq 1} T_{k_2} \frac{z^{k_2}}{k_2!} \cdots \sum_{k_r \geq 1} T_{k_r} \frac{z^{k_r}}{k_r!} \\ &= v \sum_{r \geq 1} r \varphi_r N(z, v) (T(z))^{r-1} = v \varphi'(T(z)) N(z, v). \end{aligned} \quad (13)$$

Thus we get by combining (12) and (14) the following first order linear differential equation for the generating function $N(z, v)$:

$$\frac{\partial}{\partial z} N(z, v) = v \varphi'(T(z)) N(z, v), \quad (15)$$

with initial condition $N(0, v) = T_1 = \varphi_0$. Equation (15) has the general solution

$$N(z, v) = C(v) \exp \left\{ v \int_0^z \varphi'(T(t)) dt \right\},$$

¹ v is now a complex variable, not the name of a node.

with a function $C(v)$. Evaluating at $z = 0$ and adapting to the initial condition gives now $C(v) = \varphi_0$ and thus the solution

$$N(z, v) = \varphi_0 \exp \left\{ v \int_0^z \varphi'(T(t)) dt \right\}. \quad (16)$$

Using the relation $T'(z) = \varphi(T(z))$ we can simplify the above expression (16):

$$\begin{aligned} N(z, v) &= \varphi_0 \exp \left\{ v \int_0^z \frac{\varphi'(T(t))T'(t)}{\varphi(T(t))} dt \right\} \\ &= \varphi_0 \exp \left\{ v \int_0^z (\log \varphi(T(t)))' dt \right\} \\ &= \varphi_0 \exp \left\{ v(\log \varphi(T(z)) - \log \varphi_0) \right\} \end{aligned}$$

and thus obtain the following explicit formula for the generating function $N(z, v)$:

$$N(z, v) = \varphi_0 \left(\frac{\varphi(T(z))}{\varphi_0} \right)^v. \quad (17)$$

4.2. Reproving results for the most interesting tree families. From this explicit solution (17) we can now easily obtain, by specializing $\varphi(t)$, the exact probabilities $\mathbb{P}\{D_n = m\}$ that the level of the largest node is exactly m for the 3 tree families of recursive trees, heap ordered trees, and binary increasing trees. Of course, these results are well known, but were previously shown via different methods.

- *Recursive trees.* We get by using (3) and (17) the generating function

$$N(z, v) = \frac{1}{(1-z)^v}, \quad (18)$$

which leads by extracting coefficients via (11) to the probabilities

$$\mathbb{P}\{D_n = m\} = \frac{(n-1)!}{T_n} [z^{n-1}v^m] \frac{1}{(1-z)^v} = \frac{1}{(n-1)!} \begin{bmatrix} n-1 \\ m \end{bmatrix}. \quad (19)$$

Equation (18) yields via singularity analysis also the following asymptotic expansion of the probability generating function:

$$\sum_{m \geq 0} \mathbb{P}\{D_n = m\} v^m = [z^{n-1}] \frac{1}{(1-z)^v} = \frac{n^{v-1}}{\Gamma(v)} (1 + \mathcal{O}(n^{-1+\epsilon})), \quad (20)$$

uniformly in a neighbourhood of $v = 1$ with an arbitrary small $\epsilon > 0$. When substituting $v = e^s$ in (20) we obtain a uniform expansion of the moment generating function of D_n , and a direct application of the quasi power theorem (Theorem 4) shows the Gaussian limiting distribution result for D_n with mean resp. variance

$$\mathbb{E}(D_n) = \log n + \mathcal{O}(1) \quad \text{and} \quad \mathbb{V}(D_n) = \log n + \mathcal{O}(1).$$

The exact formula (19) was given first in [13] and the limiting distribution result was shown independently in [2] and [7].

Of course, one can compute expectation and variance *exactly*, viz.

$$\mathbb{E}(D_n) = H_{n-1} \quad \text{and} \quad \mathbb{V}(D_n) = H_{n-1} - H_{n-1}^{(2)}.$$

- *Heap ordered trees.* We get by using (4) and (17)

$$N(z, v) = \frac{1}{(1 - 2z)^{v/2}}, \quad (21)$$

and with (11) the probabilities

$$\mathbb{P}\{D_n = m\} = \frac{(n-1)!}{T_n} [z^{n-1}v^m] \frac{1}{(1-2z)^{v/2}} = \frac{2^{n-1-m}}{(2n-3)!!} \begin{bmatrix} n-1 \\ m \end{bmatrix}. \quad (22)$$

Equation (21) leads to the asymptotic expansion

$$\sum_{m \geq 0} \mathbb{P}\{D_n = m\} v^m = \frac{\sqrt{\pi}}{\Gamma(\frac{v}{2})} n^{\frac{v-1}{2}} (1 + \mathcal{O}(n^{-1+\epsilon})), \quad (23)$$

uniformly in a neighbourhood of $v = 1$ with an arbitrary small $\epsilon > 0$. Again an application of the quasi power theorem to the corresponding moment generating function shows the Gaussian limiting distribution result for D_n with mean resp. variance

$$\mathbb{E}(D_n) = \frac{1}{2} \log n + \mathcal{O}(1) \quad \text{and} \quad \mathbb{V}(D_n) = \frac{1}{2} \log n + \mathcal{O}(1).$$

Both, the exact formula (22) and the limiting distribution result were obtained in [9]. Explicit versions of expectation and variance are

$$\begin{aligned} \mathbb{E}(D_n) &= H_{2n-2} - \frac{1}{2} H_{n-1}, \\ \mathbb{V}(D_n) &= H_{2n-2} - \frac{1}{2} H_{n-1} - H_{2n-2}^{(2)} + \frac{1}{4} H_{n-1}^{(2)}. \end{aligned}$$

- *Binary increasing trees.* We obtain by using (5) and (17)

$$N(z, v) = \frac{1}{(1-z)^{2v}}, \quad (24)$$

and via (11)

$$\mathbb{P}\{D_n = m\} = \frac{2^m}{n!} \begin{bmatrix} n-1 \\ m \end{bmatrix}. \quad (25)$$

Equation (24) gives the following asymptotic expansion

$$\sum_{m \geq 0} \mathbb{P}\{D_n = m\} v^m = \frac{n^{2(v-1)}}{\Gamma(2v)} (1 + \mathcal{O}(n^{-1+\epsilon})), \quad (26)$$

uniformly in a neighbourhood of $v = 1$ with an arbitrary small $\epsilon > 0$. The quasi power theorem shows now after substituting $v = e^s$ the Gaussian limiting distribution result for D_n with mean resp. variance

$$\mathbb{E}(D_n) = 2 \log n + \mathcal{O}(1) \quad \text{and} \quad \mathbb{V}(D_n) = 2 \log n + \mathcal{O}(1).$$

The exact formula (25) and the limiting distribution result are given in [6] resp. [8]. Explicit versions of expectation and variance are

$$\mathbb{E}(D_n) = 2H_n - 2 \quad \text{and} \quad \mathbb{V}(D_n) = 2H_n + 2 - 4H_n^{(2)}.$$

4.3. Limiting distribution results for polynomial increasing tree families. In the following, we only work out the case of an aperiodic degree-weight generating function $\varphi(t)$, i. e., period $q = 1$, but the general case $q \geq 2$ is fully analogous, where the contributions of the q dominant singularities have to be added.

We start with the following asymptotic expansion around the dominant singularity $z = \rho$ as given by (6), which is obtained from (8):

$$T'(z) = \frac{\delta}{\rho\eta} \frac{1}{(1 - \frac{z}{\rho})^{\delta+1}} \left(1 + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)^{2\delta}\right)\right). \quad (27)$$

With (27) we get then from (17) the expansion

$$\begin{aligned} N(z, v) &= \varphi_0 \left(\frac{T'(z)}{\varphi_0} \right)^v = \varphi_0 \exp \left(v \log \left(\frac{\delta}{\rho\eta\varphi_0} \frac{1}{(1 - \frac{z}{\rho})^{\delta+1}} \left(1 + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)^{2\delta}\right)\right) \right) \right) \\ &= \varphi_0 \left(\frac{\delta}{\rho\eta\varphi_0} \right)^v \frac{1}{(1 - \frac{z}{\rho})^{v(\delta+1)}} \left(1 + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)^{2\delta}\right)\right), \end{aligned} \quad (28)$$

uniformly in a neighbourhood of $v = 1$.

We further get from (28) and (11) by using singularity analysis the expansion

$$\begin{aligned} \sum_{m \geq 0} \mathbb{P}\{D_n = m\} v^m &= \frac{(n-1)!}{T_n} [z^{n-1}] N(z, v) \\ &= \frac{(n-1)!}{T_n} \varphi_0 \left(\frac{\delta}{\rho\eta\varphi_0} \right)^v \frac{\rho^{-n+1} n^{v(\delta+1)-1}}{\Gamma(v(\delta+1))} \left(1 + \mathcal{O}(n^{-2\delta+\epsilon})\right), \end{aligned}$$

uniformly in a neighbourhood of $v = 1$ with an arbitrary small $\epsilon > 0$. Together with the asymptotic expansion (9) of the total weights T_n this gives the following asymptotic expansion of the probability generating function:

$$\sum_{m \geq 0} \mathbb{P}\{D_n = m\} v^m = \frac{\varphi_0 \eta \rho \Gamma(\delta)}{\Gamma(v(\delta+1))} \left(\frac{\delta}{\rho\eta\varphi_0} \right)^v e^{(v-1)(\delta+1) \log n} \left(1 + \mathcal{O}(n^{-2\delta+\epsilon})\right). \quad (29)$$

An application of the quasi power theorem leads after substituting $v = e^s$ in (29) immediately to the Gaussian limiting distribution result for D_n with expectation resp. variance

$$\mathbb{E}(D_n) = (\delta + 1) \log n + \mathcal{O}(1), \quad \text{and} \quad \mathbb{V}(D_n) = (\delta + 1) \log n + \mathcal{O}(1),$$

where δ is given by (7).

5. THE LEVEL OF ARBITRARY NODES

5.1. Treating the recurrence. Analogously to Subsection 4.1, we will obtain from the formal recursive description (2), a recurrence for the probabilities $\mathbb{P}\{D_{n,j} = m\}$, where $D_{n,j}$ is the random variable that counts the level of node j in a random increasing tree of size n . For increasing trees of size n with root-degree r and subtrees with sizes k_1, \dots, k_r , enumerated from left to right, where the node labelled by j lies in the leftmost subtree and is the i -th smallest node in this subtree, we can reduce the computation of the probabilities $\mathbb{P}\{D_{n,j} = m\}$ to the probabilities $\mathbb{P}\{D_{k_1,i} = m-1\}$. We get as factor the total weight of the r subtrees and the root node $\varphi_r T_{k_1} \cdots T_{k_r}$, divided by the total weight T_n of trees of size n and multiplied by the number of order preserving relabellings of the r subtrees, which are given here by

$$\binom{j-2}{i-1} \binom{n-j}{k_1-i} \binom{n-1-k_1}{k_2, k_3, \dots, k_r} :$$

the $i-1$ labels smaller than j are chosen from $2, 3, \dots, j-1$, the k_1-i labels larger than j are chosen from $j+1, \dots, n$, and the remaining $n-1-k_1$ labels are distributed to the second, third, \dots , r -th subtree. Again due to symmetry arguments we obtain a factor r , if the node j is the i -th smallest node in the second, third, \dots , r -th subtree. Summing up over all choices for the rank i of label j in its subtree, the subtree sizes k_1, \dots, k_r , and the degree r of the root node gives the following recurrence (30).

$$\begin{aligned} \mathbb{P}\{D_{n,j} = m\} &= \sum_{r \geq 1} r \varphi_r \sum_{\substack{k_1 + \dots + k_r = n-1, \\ k_1, \dots, k_r \geq 1}} \frac{T_{k_1} \cdots T_{k_r}}{T_n} \times \\ &\times \sum_{i=1}^{\min\{k_1, j-1\}} \mathbb{P}\{D_{k_1,i} = m-1\} \binom{j-2}{i-1} \binom{n-j}{k_1-i} \binom{n-1-k_1}{k_2, k_3, \dots, k_r}, \end{aligned} \quad (30)$$

for $n \geq j \geq 2$. For $j = 1$ we obtain $\mathbb{P}\{D_{n,1} = m\} = \delta_{m,0}$.

To treat this recurrence (30) we set $n := k + j$ with $k \geq 0$ and define the trivariate generating function

$$N(z, u, v) := \sum_{k \geq 0} \sum_{j \geq 1} \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\} T_{k+j} \frac{z^{j-1}}{(j-1)!} \frac{u^k}{k!} v^m. \quad (31)$$

Multiplying (30) with $T_{k+j} \frac{z^{j-2}}{(j-2)!} \frac{u^k}{k!} v^m$ and summing up over $k \geq 0$, $j \geq 2$ and $m \geq 0$ leads then to

$$\sum_{k \geq 0} \sum_{j \geq 2} \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\} T_{k+j} \frac{z^{j-2}}{(j-2)!} \frac{u^k}{k!} v^m = \frac{\partial}{\partial z} N(z, u, v), \quad (32)$$

and

$$\begin{aligned}
\mathbf{RS} &:= \sum_{k \geq 0} \sum_{j \geq 2} \sum_{m \geq 0} \sum_{r \geq 1} r \varphi_r \sum_{\substack{k_1 + \dots + k_r = k + j - 1, \\ k_1, \dots, k_r \geq 1}} \frac{T_{k_1} \cdots T_{k_r}}{T_{k+j}} \times \\
&\times \sum_{i=1}^{\min\{k_1, j-1\}} \mathbb{P}\{D_{k_1, i} = m - 1\} \binom{j-2}{i-1} \binom{k}{k_1 - i} \binom{k+j-1-k_1}{k_2, k_3, \dots, k_r} T_{k+j} \frac{z^{j-2}}{(j-2)!} \frac{u^k}{k!} v^m \\
&= \sum_{k \geq 0} \sum_{j \geq 2} \sum_{m \geq 0} \sum_{r \geq 1} r \varphi_r \sum_{\substack{k_1 + \dots + k_r = k + j - 1, \\ k_1, \dots, k_r \geq 1}} T_{k_1} \cdots T_{k_r} \times \\
&\times \sum_{i=1}^{\min\{k_1, j-1\}} \mathbb{P}\{D_{k_1, i} = m - 1\} \frac{(k+j-1-k_1)! z^{j-2} u^k v^m}{(i-1)!(j-i-1)!(k_1-i)!(k-k_1+i)!k_2!k_3! \cdots k_r!} \\
&= \sum_{r \geq 1} r \varphi_r \sum_{m \geq 0} \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} \cdots \sum_{k_r \geq 1} \sum_{j \geq 1} T_{k_1} \cdots T_{k_r} \\
&\times \sum_{i=1}^{\min\{k_1, j-1\}} \frac{\mathbb{P}\{D_{k_1, i} = m - 1\} (k_2 + k_3 + \dots + k_r)! z^{j-2} u^{k_1+k_2+\dots+k_r-j+1} v^m}{(i-1)!(j-i-1)!(k_1-i)!k_2!k_3! \cdots k_r!(k_2+k_3+\dots+k_r-j+1+i)!}
\end{aligned}$$

where we use in the last equation that $k = k_1 + \dots + k_r - j + 1$. We set now $k_1 = i + l$ with $l \geq 0$ and obtain after a change of variables

$$\begin{aligned}
\mathbf{RS} &= \sum_{r \geq 1} r \varphi_r \sum_{m \geq 0} \sum_{i \geq 1} \sum_{l \geq 0} \sum_{k_2 \geq 1} \sum_{k_3 \geq 1} \cdots \sum_{k_r \geq 1} \sum_{j \geq i+1} T_{i+l} T_{k_2} \cdots T_{k_r} \mathbb{P}\{D_{i+l, i} = m - 1\} \times \\
&\times \frac{(k_2 + \dots + k_r)! z^{j-2} u^{i-j+1+l+k_2+\dots+k_r} v^m}{(i-1)!(j-i-1)!l!k_2! \cdots k_r!(k_2 + \dots + k_r - j + i + 1)!} \\
&= v \sum_{l \geq 0} \sum_{i \geq 1} \sum_{m \geq 1} \mathbb{P}\{D_{i+l, i} = m - 1\} T_{i+l} \frac{z^{i-1}}{(i-1)!} \frac{u^l}{l!} v^{m-1} \sum_{r \geq 1} r \varphi_r \sum_{k_2 \geq 1} \cdots \sum_{k_r \geq 1} \\
&\times \sum_{j \geq i+1} \frac{T_{k_2} \cdots T_{k_r} (k_2 + \dots + k_r)! z^{j-i-1} u^{k_2+\dots+k_r-(j-i-1)}}{(j-i-1)!k_2! \cdots k_r!(k_2 + \dots + k_r - (j-i-1))!} \\
&\stackrel{s:=j-i-1}{=} v N(z, u, v) \sum_{r \geq 1} r \varphi_r \sum_{k_2 \geq 1} \frac{T_{k_2}}{k_2!} \cdots \sum_{k_r \geq 1} \frac{T_{k_r}}{k_r!} \sum_{s \geq 0} \binom{k_2 + \dots + k_r}{s} z^s u^{k_2+\dots+k_r-s} \\
&= v N(z, u, v) \sum_{r \geq 1} r \varphi_r \sum_{k_2 \geq 1} \frac{T_{k_2}}{k_2!} \cdots \sum_{k_r \geq 1} \frac{T_{k_r}}{k_r!} (z + u)^{k_2+\dots+k_r} \\
&= v N(z, u, v) \sum_{r \geq 1} r \varphi_r (T(z + u))^{r-1} = v \varphi'(T(z + u)) N(z, u, v). \tag{33}
\end{aligned}$$

Combining the expressions (32) and (33) gives then the following first order linear differential equation for the generating functions $N(z, u, v)$:

$$\frac{\partial}{\partial z} N(z, u, v) = v \varphi'(T(z + u)) N(z, u, v). \tag{34}$$

The general solution of (34) is then given by

$$N(z, u, v) = C(u, v) \exp \left\{ v \int_0^z \varphi'(T(t+u)) dt \right\},$$

with a function $C(u, v)$. Since

$$\begin{aligned} N(0, u, v) &= \sum_{k \geq 0} \sum_{m \geq 0} \mathbb{P}\{D_{k+1,1} = m\} T_{k+1} \frac{u^k}{k!} v^m \\ &= \sum_{k \geq 0} T_{k+1} \frac{u^k}{k!} = T'(u) = \varphi(T(u)), \end{aligned}$$

we obtain the required solution

$$N(z, u, v) = \varphi(T(u)) \exp \left\{ v \int_0^z \varphi'(T(t+u)) dt \right\},$$

which can again be simplified by using $T'(z) = \varphi(T(z))$. We get thus the following exact formula for the trivariate generating function $N(z, u, v)$:

$$N(z, u, v) = \varphi(T(u)) \left(\frac{\varphi(T(z+u))}{\varphi(T(u))} \right)^v = T'(u) \left(\frac{T'(z+u)}{T'(u)} \right)^v. \quad (35)$$

5.2. Proving results for the most interesting tree families. From this generating function solution (35) it is again an easy task to extract coefficients for recursive trees, heap ordered trees and binary increasing trees and compute the probabilities $\mathbb{P}\{D_{n,j} = m\}$.

- *Recursive trees.* Here we obtain

$$N(z, u, v) = \frac{1}{1-u} \left(\frac{1-u}{1-(u+z)} \right)^v = \frac{1}{(1-u) \left(1 - \frac{z}{1-u}\right)^v} \quad (36)$$

and the probability generating function

$$\begin{aligned} \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\} v^m &= \frac{(j-1)!k!}{T_{k+j}} [z^{j-1}u^k] N(z, u, v) \\ &= \frac{(j-1)!k!}{(k+j-1)!} \binom{v+j-2}{j-1} [u^k] \frac{1}{(1-u)^j} \\ &= \frac{(j-1)!k!}{(k+j-1)!} \binom{v+j-2}{j-1} \binom{k+j-1}{k} = \binom{v+j-2}{j-1}. \end{aligned} \quad (37)$$

Therefore the probability generating function and thus the distribution of $D_{k+j,j}$ is independent of k and one gets

$$\mathbb{P}\{D_{n,j} = m\} = \mathbb{P}\{D_j = m\} = \frac{1}{(j-1)!} \begin{bmatrix} j-1 \\ m \end{bmatrix}.$$

Expectation and variance are given by

$$\mathbb{E}(D_{n,j}) = H_{j-1} \quad \text{and} \quad \mathbb{V}(D_{n,j}) = H_{j-1} - H_{j-1}^{(2)}.$$

- *Heap ordered trees.* Here we obtain

$$N(z, u, v) = \frac{(1-2u)^{v/2}}{\sqrt{1-2u}(1-2(u+z))^{v/2}} = \frac{1}{\sqrt{1-2u}\left(1-\frac{2z}{1-2u}\right)^{v/2}} \quad (38)$$

and the probability generating function

$$\sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m = \frac{(j-1)!k!}{T_{k+j}} 2^{k+j-1} \binom{\frac{v}{2}+j-2}{j-1} \binom{k+j-\frac{3}{2}}{k},$$

which simplifies to

$$\begin{aligned} \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m &= \frac{(j-1)!k! \binom{\frac{v}{2}+j-2}{j-1} \binom{k+j-\frac{3}{2}}{k}}{(k+j-1)! \binom{k+j-\frac{3}{2}}{k+j-1}} \\ &= \frac{\binom{\frac{v}{2}+j-2}{j-1}}{\binom{j-\frac{3}{2}}{j-1}} = \prod_{i=1}^{j-1} \frac{2i-2+v}{2i-1}. \end{aligned} \quad (39)$$

Again we find that the distribution of $D_{k+j,j}$ is independent of k and one gets

$$\mathbb{P}\{D_{n,j} = m\} = \mathbb{P}\{D_j = m\} = \frac{2^{j-1-m}}{(2j-3)!!} \begin{bmatrix} j-1 \\ m \end{bmatrix}.$$

Expectation and variance are given by

$$\begin{aligned} \mathbb{E}(D_{n,j}) &= H_{2j-2} - \frac{1}{2}H_{j-1}, \\ \mathbb{V}(D_{n,j}) &= H_{2j-2} - \frac{1}{2}H_{j-1} - H_{2j-2}^{(2)} + \frac{1}{4}H_{j-1}^{(2)}. \end{aligned}$$

- *Binary increasing trees.* Here we get

$$N(z, u, v) = \frac{1}{(1-u)^2} \frac{(1-u)^{2v}}{(1-(u+z))^{2v}} = \frac{1}{(1-u)^2 \left(1-\frac{z}{1-u}\right)^{2v}}, \quad (40)$$

and the probability generating function

$$\sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m = \frac{(j-1)!k!}{(k+j)!} \binom{2v+j-2}{j-1} \binom{k+j}{k} = \frac{1}{j} \binom{2v+j-2}{j-1}. \quad (41)$$

The distribution of $D_{k+j,j}$ is also independent of k and one gets

$$\mathbb{P}\{D_{n,j} = m\} = \mathbb{P}\{D_j = m\} = \frac{2^m}{j!} \begin{bmatrix} j-1 \\ m \end{bmatrix}.$$

Expectation and variance are given by

$$\mathbb{E}(D_{n,j}) = 2H_j - 2 \quad \text{and} \quad \mathbb{V}(D_{n,j}) = 2H_j + 2 - 4H_j^{(2)}.$$

5.3. Depth of the $(k + 1)$ -st largest node in polynomial increasing tree families.

We study now for polynomial families of increasing trees the random variable $D_{n,j}$ for the case $n = j + k$ with k fixed, or in other words we study the depth of the $(k + 1)$ -st largest node in a random tree of size n for $n \rightarrow \infty$. Again we work out only the case for aperiodic degree-weight generating functions $\varphi(t)$.

To extract coefficients $[z^{j-1}u^k]$ from $N(z, u, v)$ as given by (35) for k fixed we will first expand $(T'(z + u))^v$ around $u = 0$ and obtain

$$\begin{aligned} (T'(z + u))^v &= \left(\sum_{i \geq 0} T^{(i+1)}(z) \frac{u^i}{i!} \right)^v = (T'(z))^v \left(1 + \sum_{i \geq 1} \frac{T^{(i+1)}(z)}{T'(z)} \frac{u^i}{i!} \right)^v \\ &= (T'(z))^v \sum_{l \geq 0} \binom{v}{l} \left(\sum_{i \geq 1} \frac{T^{(i+1)}(z)}{T'(z)} \frac{u^i}{i!} \right)^l, \end{aligned}$$

and thus

$$[u^k](T'(z + u))^v = [u^k](T'(z))^v \sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{T^{(i+1)}(z)}{T'(z)} \frac{u^i}{i!} \right)^l, \quad (42)$$

since the other terms give no contribution.

From (8) we get the local expansions

$$T^{(i+1)}(z) = \frac{\delta^{\overline{i+1}}}{\eta \rho^{i+1} (1 - \frac{z}{\rho})^{\delta+i+1}} \left(1 + \mathcal{O}((1 - \frac{z}{\rho})^{2\delta}) \right), \quad (43)$$

in a neighbourhood of the dominant singularity $z = \rho$, which hold for $i \geq 0$. Of course the \mathcal{O} -bound is not uniform for all $i \in \mathbb{N}$, but we require only a common bound for $i \leq k$ with k fixed and this exists. We obtain thus by using (43) for k fixed the expansion

$$\begin{aligned} (T'(z))^v &\sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{T^{(i+1)}(z)}{T'(z)} \frac{u^i}{i!} \right)^l \\ &= \left(\frac{\delta}{\eta \rho} \right)^v \frac{1}{(1 - \frac{z}{\rho})^{v(\delta+1)}} \left(1 + \mathcal{O}((1 - \frac{z}{\rho})^{2\delta}) \right) \times \\ &\quad \times \sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{\delta^{\overline{i+1}} \rho (1 - \frac{z}{\rho})^{\delta+1}}{\delta \rho^{i+1} (1 - \frac{z}{\rho})^{\delta+i+1}} \frac{u^i}{i!} \left(1 + \mathcal{O}((1 - \frac{z}{\rho})^{2\delta}) \right) \right)^l \\ &= \left(\frac{\delta}{\eta \rho} \right)^v \frac{1}{(1 - \frac{z}{\rho})^{v(\delta+1)}} \sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{(\delta + 1)^{\overline{i}}}{\rho^i (1 - \frac{z}{\rho})^i} \frac{u^i}{i!} \right)^l \left(1 + \mathcal{O}((1 - \frac{z}{\rho})^{2\delta}) \right), \end{aligned}$$

and thus from (42):

$$\begin{aligned} [u^k](T'(z + u))^v &= \\ &\left([u^k] \left(\frac{\delta}{\eta \rho} \right)^v \frac{1}{(1 - \frac{z}{\rho})^{v(\delta+1)}} \sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{(\delta + 1)^{\overline{i}}}{\rho^i (1 - \frac{z}{\rho})^i} \frac{u^i}{i!} \right)^l \right) \left(1 + \mathcal{O}((1 - \frac{z}{\rho})^{2\delta}) \right). \quad (44) \end{aligned}$$

It remains to extract coefficients from the main term of (44), where we can use

$$\begin{aligned}\Xi &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \sum_{l=0}^k \binom{v}{l} \left(\sum_{i=1}^k \frac{(\delta+1)^{\bar{i}} u^i}{\rho^i \left(1 - \frac{z}{\rho}\right)^i i!} \right)^l \\ &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \sum_{l \geq 0} \binom{v}{l} \left(\sum_{i \geq 1} \frac{(\delta+1)^{\bar{i}} u^i}{\rho^i \left(1 - \frac{z}{\rho}\right)^i i!} \right)^l,\end{aligned}$$

since the remaining summands do not give contributions. Now we can sum up and obtain

$$\begin{aligned}\Xi &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \sum_{l \geq 0} \binom{v}{l} \left(\sum_{i \geq 1} \frac{(\delta+1)^{\bar{i}} u^i}{\rho^i \left(1 - \frac{z}{\rho}\right)^i i!} \right)^l \\ &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \left(1 + \sum_{i \geq 1} \frac{(\delta+1)^{\bar{i}} u^i}{\rho^i \left(1 - \frac{z}{\rho}\right)^i i!} \right)^v \\ &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \left(\sum_{i \geq 0} \binom{\delta+i}{i} \left(\frac{u}{\rho \left(1 - \frac{z}{\rho}\right)} \right)^i \right)^v \\ &= [u^k] \left(\frac{\delta}{\eta\rho} \right)^v \frac{1}{\left(1 - \frac{z}{\rho}\right)^{v(\delta+1)}} \frac{1}{\left(1 - \frac{u}{\rho \left(1 - \frac{z}{\rho}\right)}\right)^{v(\delta+1)}} \\ &= \left(\frac{\delta}{\eta\rho} \right)^v \rho^{-k} \binom{k + v(\delta+1) - 1}{k} \frac{1}{\left(1 - \frac{z}{\rho}\right)^{k+v(\delta+1)}}.\end{aligned}\tag{45}$$

Equations (44) and (45) give thus the required local expansion

$$[u^k] (T'(z+u))^v = \left(\frac{\delta}{\eta\rho} \right)^v \rho^{-k} \binom{k + v(\delta+1) - 1}{k} \frac{1}{\left(1 - \frac{z}{\rho}\right)^{k+v(\delta+1)}} \left(1 + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)^{2\delta}\right) \right),\tag{46}$$

in a neighbourhood of $z = \rho$, and also as a consequence

$$\begin{aligned}[u^k] N(z, u, v) &= \sum_{r=0}^k [u^r] (T'(z+u))^v [u^{k-r}] (T'(u))^{-v+1} \\ &= [u^k] (T'(z+u))^v [u^0] (T'(u))^{-v+1} + \mathcal{O}([u^{k-1}] (T'(z+u))^v) \\ &= \left(\frac{\delta}{\eta\rho} \right)^v \varphi_0^{-v+1} \rho^{-k} \binom{k + v(\delta+1) - 1}{k} \frac{1}{\left(1 - \frac{z}{\rho}\right)^{k+v(\delta+1)}} \left(1 + \mathcal{O}\left(\left(1 - \frac{z}{\rho}\right)^{2\delta}\right) \right).\end{aligned}\tag{47}$$

Now singularity analysis gives from (47)

$$\begin{aligned}\sum_{m \geq 0} \mathbb{P}\{D_{k+j, j} = m\} v^m &= \frac{(j-1)! k!}{T_{k+j}} [z^{j-1} u^k] N(z, u, v) \\ &= \frac{(j-1)! k!}{T_{k+j}} \left(\frac{\delta}{\eta\rho} \right)^v \varphi_0^{-v+1} \rho^{-k} \binom{k + v(\delta+1) - 1}{k} \rho^{-j+1} \frac{j^{k+v(\delta+1)-1}}{\Gamma(k + v(\delta+1))} \left(1 + \mathcal{O}(j^{-2\delta+\epsilon}) \right),\end{aligned}$$

for an arbitrary small $\epsilon > 0$. By using (9) we get then the following asymptotic expansion of the probability generating function:

$$\begin{aligned}
& \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\} v^m \\
&= \frac{(j-1)! k! \eta \Gamma(\delta) \rho^{k+j} (k+j)^{1-\delta} \varphi_0^{-v+1}}{(k+j)! \rho^{k+j-1} \Gamma(k+v(\delta+1))} \left(\frac{\delta}{\eta \rho}\right)^v \binom{k+v(\delta+1)-1}{k} j^{k+v(\delta+1)-1} (1 + \mathcal{O}(j^{-2\delta+\epsilon})) \\
&= \frac{k! \eta \rho \Gamma(\delta) \varphi_0^{-v+1}}{\Gamma(k+v(\delta+1))} \left(\frac{\delta}{\eta \rho}\right)^v \binom{k+v(\delta+1)-1}{k} j^{(v-1)(\delta+1)} (1 + \mathcal{O}(j^{-2\delta+\epsilon})) \\
&= \frac{\eta \rho \varphi_0 \Gamma(\delta)}{\Gamma(v(\delta+1))} \left(\frac{\delta}{\varphi_0 \eta \rho}\right)^v j^{(v-1)(\delta+1)} (1 + \mathcal{O}(j^{-2\delta+\epsilon})). \tag{48}
\end{aligned}$$

After substituting $v = e^s$, the Gaussian limiting distribution result follows from this expansion (48), which holds uniformly in a neighbourhood of $v = 1$, i. e., $s = 0$, by an application of the quasi power theorem.

6. VERY SIMPLE INCREASING TREE FAMILIES

6.1. Increasing trees with a randomness preserving property. As we saw in Subsection 5.2, there are several important increasing tree families, where the distribution of the depth $D_{n,j}$ of node j does not depend on the size $n \geq j$ of the tree, that means $D_{n,j} = D_j$. In order to find such increasing tree families we consider families \mathcal{T} with the property that when starting with a random tree of \mathcal{T} of size n and removing all nodes larger than j , we obtain a random tree of \mathcal{T} of size j . By iterating the argument, it is sufficient to show that after removing node n in a random size- n tree we get a random tree of \mathcal{T} of size $n-1$. This randomness preserving property can be described easily via the equation

$$\frac{w(T')}{w(T'')} = \frac{\sum_{T: T \xrightarrow{(n)} T'} w(T)}{\sum_{T: T \xrightarrow{(n)} T''} w(T)}, \tag{49}$$

which must hold for all ordered trees T', T'' of size $|T'| = |T''| = n-1$ and ordered trees T of size $|T| = n$. Here, $T \xrightarrow{(n)} T'$ describes the fact that by cutting off node n from the tree T we get the tree T' . We assume now that T' is obtained from T by cutting off node n , which was originally attached at node v . If $d(v)$ is the out-degree of node v in the tree T' , there are $d(v)+1$ different trees T that lead to the same tree T' when cutting off node n . We obtain then from (49) the equivalent characterization

$$\frac{\prod_{v \in T'} \varphi_{d(v)}}{\prod_{v \in T''} \varphi_{d(v)}} = \frac{\prod_{v \in T'} \varphi_{d(v)} \sum_{v \in T'} \frac{(d(v)+1) \varphi_{d(v)+1}}{\varphi_{d(v)}}}{\prod_{v \in T''} \varphi_{d(v)} \sum_{v \in T''} \frac{(d(v)+1) \varphi_{d(v)+1}}{\varphi_{d(v)}}}, \tag{50}$$

and thus the condition

$$\sum_{v \in T'} \frac{(d(v)+1) \varphi_{d(v)+1}}{\varphi_{d(v)}} = \sum_{v \in T''} \frac{(d(v)+1) \varphi_{d(v)+1}}{\varphi_{d(v)}}, \tag{51}$$

for all ordered trees T', T'' of size $|T'| = |T''| = n - 1$. This condition (51) was already analysed in [12], where a complete characterization of degree-weight generating functions $\varphi(t)$ satisfying this property was given in a slightly different context. As already pointed out in [5], this characterization of $\varphi(t)$ is equivalent to the characterization of $\varphi(t)$ satisfying property (1) in Lemma 5. It turns out that there are only three possible cases, which are given in Lemma 5.

6.2. Construction of increasing trees via an insertion process. Now we want to study increasing tree families which can be constructed via an *insertion process* or a *probabilistic growth rule*. It is well known that recursive trees, heap ordered trees, binary increasing trees, and more generally d -ary increasing trees, which are used as polymerization model in chemistry (see [10]) satisfy this property.

As already pointed out in the introduction we consider increasing tree families \mathcal{T} , which have the property that for every tree T' of size $n - 1$ with vertices v'_1, \dots, v'_{n-1} there exist probabilities $p_{T'}(v'_1), \dots, p_{T'}(v'_{n-1})$, such that when starting with a random tree T' of size $n - 1$, choosing a vertex v'_i in T' according to the probabilities $p_{T'}(v'_i)$, i. e., $\sum_{i=1}^{n-1} p_{T'}(v'_i) = 1$, and attaching node n to it at one of the $d(v'_i) + 1$ possible positions (which must be all equally likely due to the “symmetric” recursive description of increasing tree families), we obtain a random increasing tree T of the family \mathcal{T} of size n .

To formulate this property we assume that we have given trees T', T'' of size $|T'| = |T''| = n - 1$ with $w(T') = \prod_{v \in T'} \varphi_{d(v)}$ and $w(T'') = \prod_{v \in T''} \varphi_{d(v)}$, nodes v'_1, \dots, v'_{n-1} resp. v''_1, \dots, v''_{n-1} with probabilities $p_{T'}(v'_1), \dots, p_{T'}(v'_{n-1})$ resp. $p_{T''}(v''_1), \dots, p_{T''}(v''_{n-1})$, such that $\sum_{i=1}^{n-1} p_{T'}(v'_i) = \sum_{i=1}^{n-1} p_{T''}(v''_i) = 1$.

When attaching node n to vertex $v'_k \in T'$ resp. vertex $v''_l \in T''$ to one of the $d_{v'_k} + 1$ resp. $d_{v''_l} + 1$ positions we obtain the trees T'_k resp. T''_l , which have the weights

$$w(T'_k) = \varphi_0 \frac{\varphi_{d(v'_k)+1}}{\varphi_{d(v'_k)}} \prod_{v \in T'} \varphi_{d(v)}, \quad \text{and} \quad w(T''_l) = \varphi_0 \frac{\varphi_{d(v''_l)+1}}{\varphi_{d(v''_l)}} \prod_{v \in T''} \varphi_{d(v)}. \quad (52)$$

On the one hand when starting with random increasing trees T' resp. T'' of size $|T'| = |T''| = n - 1$, i. e., chosen with probability proportional to their weights, we obtain the following probabilities that the trees T'_k resp. T''_l are obtained by the insertion process:

$$\tilde{w}(T'_k) = \frac{p_{T'}(v'_k)}{d(v'_k) + 1} \frac{\prod_{v \in T'} \varphi_{d(v)}}{T_{n-1}} \quad \text{and} \quad \tilde{w}(T''_l) = \frac{p_{T''}(v''_l)}{d(v''_l) + 1} \frac{\prod_{v \in T''} \varphi_{d(v)}}{T_{n-1}}. \quad (53)$$

Since the resulting trees T'_k and T''_l must be random increasing trees of \mathcal{T} of size n it must hold that

$$\frac{w(T'_k)}{w(T''_l)} = \frac{\tilde{w}(T'_k)}{\tilde{w}(T''_l)},$$

and thus

$$\frac{\varphi_0 \frac{\varphi_{d(v'_k)+1}}{\varphi_{d(v'_k)}} \prod_{v \in T'} \varphi_{d(v)}}{\varphi_0 \frac{\varphi_{d(v'_l)+1}}{\varphi_{d(v'_l)}} \prod_{v \in T''} \varphi_{d(v)}} = \frac{\frac{p_{T'}(v'_k)}{d(v'_k)+1} \frac{\prod_{v \in T'} \varphi_{d(v)}}{T_{n-1}}}{\frac{p_{T''}(v'_l)}{d(v'_l)+1} \frac{\prod_{v \in T''} \varphi_{d(v)}}{T_{n-1}}},$$

or equivalently

$$\frac{1}{p_{T'}(v'_k)} \frac{(d(v'_k)+1)\varphi_{d(v'_k)+1}}{\varphi_{d(v'_k)}} = \frac{1}{p_{T''}(v'_l)} \frac{(d(v'_l)+1)\varphi_{d(v'_l)+1}}{\varphi_{d(v'_l)}} =: c(n-1), \quad (54)$$

for all trees T', T'' with size $|T'| = |T''| = n-1$, and all vertices $v'_k \in T'$ resp. $v'_l \in T''$, where $c(n)$ is a function depending only on n .

It follows from (54) further

$$\frac{(d(v'_k)+1)\varphi_{d(v'_k)+1}}{\varphi_{d(v'_k)}} = p_{T'}(v'_k) c(n-1),$$

and by summing up that

$$\sum_{v \in T'} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}} = \sum_{v \in T''} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}}, \quad (55)$$

for all trees T', T'' with size $|T'| = |T''| = n-1$. On the other hand for all tree families satisfying (55) we can define the probabilities $p_{T'}(v)$ as $\frac{1}{c(n-1)} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}}$ independently from the tree T' , and they satisfy (54). Therefore we obtain again the condition (51) and it follows that the increasing tree families, which can be constructed via an insertion process are exactly the very simple increasing tree families, which is the third equivalent property in Lemma 5.

6.3. Distribution results of the level of arbitrary nodes. It turns out that for very simple increasing tree families, which are characterized by $\varphi(t)$ given in Lemma 5, we can easily compute exact formulæ for the probability generating function $\sum_{m \geq 0} \mathbb{P}\{D_{n,j} = m\} v^m$ and the probabilities $\mathbb{P}\{D_{n,j} = m\}$ of the level of the node j in a random tree of size n (which are of course independent of n). These exact formulæ also show the Gaussian limit law of $D_{n,j}$.

We work out only the **Case C**, but the remaining cases are completely analogous. From the degree-weight generating function $\varphi(t) = \frac{\varphi_0}{(1 + \frac{c_2 t}{\varphi_0})^{-\frac{c_1}{c_2} - 1}}$, with $\varphi_0 > 0$ and $0 < -c_2 < c_1$, one easily computes either by solving the differential equation (1) or using the first property in Lemma 5 that

$$T(z) = \frac{\varphi_0}{c_2} \left(\frac{1}{(1 - c_1 z)^{\frac{c_2}{c_1}}} - 1 \right), \quad \text{and thus} \quad T'(z) = \frac{\varphi_0}{(1 - c_1 z)^{\frac{c_2}{c_1} + 1}}.$$

Therefore we obtain from (35) the generating function

$$N(z, u, v) = \varphi(T(u)) \left(\frac{\varphi(T(z+u))}{\varphi(T(u))} \right)^v = \frac{\varphi_0}{(1 - c_1 u)^{\frac{c_2}{c_1} + 1} \left(1 - \frac{c_1 z}{1 - c_1 u} \right)^{\frac{c_2}{c_1} + 1} v}. \quad (56)$$

Using

$$T_n = n![z^n]T(z) = \frac{n!\varphi_0 c_1^n}{c_2} \binom{n + \frac{c_2}{c_1} - 1}{n},$$

we get from (56) the probability generating function

$$\begin{aligned} \sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m &= \frac{(j-1)!k!}{T_{k+j}} [z^{j-1}u^k]N(z, u, v) \\ &= \frac{(j-1)!k!c_2}{(k+j)!c_1} \frac{\binom{j-2+(\frac{c_2}{c_1}+1)v}{j-1} \binom{k+j+\frac{c_2}{c_1}-1}{k}}{\binom{k+j+\frac{c_2}{c_1}-1}{k+j}} \\ &= \frac{(j-1)!c_2(\frac{c_2}{c_1}-1)!}{c_1(j+\frac{c_2}{c_1}-1)!} \binom{j-2+(\frac{c_2}{c_1}+1)v}{j-1} = \frac{(j-2+(\frac{c_2}{c_1}+1)v)^{j-1}}{(j+\frac{c_2}{c_1}-1)^{j-1}} \\ &= \frac{\prod_{i=0}^{j-2} (i+(\frac{c_2}{c_1}+1)v)}{\prod_{i=0}^{j-2} (i+\frac{c_2}{c_1}+1)} = \prod_{i=1}^{j-1} \left(1 + \frac{(c_1+c_2)(v-1)}{c_1 i + c_2}\right). \end{aligned} \quad (57)$$

From (57) we get exact formulæ for the corresponding probabilities by using the Stirling numbers of first kind:

$$\begin{aligned} \mathbb{P}\{D_{k+j,j} = m\} &= [v^m] \frac{\prod_{i=0}^{j-2} (i+(\frac{c_2}{c_1}+1)v)}{\prod_{i=0}^{j-2} (i+\frac{c_2}{c_1}+1)} = \frac{1}{(\frac{c_2}{c_1}+1)^{j-1}} [v^m] \prod_{i=0}^{j-2} \left((\frac{c_2}{c_1}+1)v + i \right) \\ &= \frac{(\frac{c_2}{c_1}+1)^m}{(\frac{c_2}{c_1}+1)^{j-1}} \left[\begin{matrix} j-1 \\ m \end{matrix} \right]. \end{aligned} \quad (58)$$

It turns out that the exact expressions (57) resp. (58) also hold for **Case A** ($c_2 = 0$), and **Case B** ($d = \frac{c_1}{c_2} + 1$).

Finally, using Stirling's expansion of the Gamma function we obtain from (57) via

$$\sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m = \frac{\Gamma(j-1+(\frac{c_2}{c_1}+1)v)\Gamma(\frac{c_2}{c_1}+1)}{\Gamma((\frac{c_2}{c_1}+1)v+1)\Gamma(j+\frac{c_2}{c_1})}$$

the asymptotic expansion

$$\sum_{m \geq 0} \mathbb{P}\{D_{k+j,j} = m\}v^m = \frac{\Gamma(\frac{c_2}{c_1}+1)}{\Gamma((\frac{c_2}{c_1}+1)v+1)} j^{(\frac{c_2}{c_1}+1)(v-1)} (1 + \mathcal{O}(j^{-1})), \quad (59)$$

which holds uniformly in a neighbourhood of $v = 1$. After the substitution $v = e^s$ in (59), an application of the quasi power theorem shows also the last part of Theorem 7.

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