THE LU-DECOMPOSITION OF LEHMER'S TRIDIAGONAL MATRIX

HELMUT PRODINGER

Dedication to George Andrews on the occasion of his 80^{th} birthday.

1. Introduction

Ekhad and Zeilberger [5] have unearthed Lehmer's [8] tridiagonal $n \times n$ matrix $M = M(n)$ with entries

$$M_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ z^{1/2} q^{(i-1)/2} & \text{if } i = j - 1, \\ z^{1/2} q^{(i-2)/2} & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lehmer [8] has computed the limit for $n \to \infty$ of the determinant of the matrix $M(n)$. Ekhad and Zeilberger [5] have generalized this result by computing the determinant of the finite matrix $M(n)$. Furthermore, a lively account of how modern computer algebra leads to a solution was given. Most prominently, the celebrated q-Zeilberger algorithm [10] and creative guessing were used.

In this note, the determinant in question is obtained by computing the LU-decomposition $LU = M$. This is done with a computer, and the exact form of L and U is obtained by guessing. A proof that this is indeed the LU-decomposition is then a routine calculation. From it, the determinant in question is computed by multiplying the diagonal elements of the matrix U. By telescoping, the final result is then quite attractive, as already stated and proved by Ekhad and Zeilberger [5].

We hope that this little contribution will be a welcome addition to the rekindled interest in Lehmer's tridiagonal determinant.

We use standard notation [1]: $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$, and the Gaussian q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$

2. The LU-decomposition of M

Let

$$\lambda(j) := \sum_{0 \le k \le j/2} \begin{bmatrix} j-k \\ k \end{bmatrix} (-1)^k q^{k(k-1)} z^k.$$

It follows from the basic recursion of the Gaussian q-binomial coefficients [1] that

$$\lambda(j) = \lambda(j-1) - zq^{j-2} \lambda(j-2). \tag{1}$$

Then we have

$$U_{j,j} = \frac{\lambda(j)}{\lambda(j-1)}, \qquad U_{j,j+1} = z^{1/2} q^{(j-1)/2},$$

and all other entries in the U-matrix are zero. Further,

$$L_{j,j} = 1, \qquad L_{j+1,j} = z^{1/2} q^{(j-1)/2} \frac{\lambda(j-1)}{\lambda(j)},$$

and all other entries in the L-matrix are zero.

The typical element of the product $(LU)_{i,j}$, that is

$$\sum_{1 \le k \le n} L_{i,k} U_{k,j}$$

is almost always zero; the exceptions are as follows: If $i = j$, then we get

$$L_{j,j} U_{j,j} + L_{j,j-1} U_{j-1,j} = \frac{\lambda(j) + z q^{j-2} \lambda(j-2)}{\lambda(j-1)} = 1,$$

because of the above recursion (1). If $i = j-1$, then we get

$$L_{j-1,j-1} U_{j-1,j} + L_{j-1,j-2} U_{j-2,j} = z^{1/2} q^{(j-2)/2},$$

and if $i = j+1$, then we get

$$L_{j+1,j+1} U_{j+1,j} + L_{j+1,j} U_{j,j} = z^{1/2} q^{(j-1)/2} \frac{\lambda(j-1)}{\lambda(j)} \frac{\lambda(j)}{\lambda(j-1)} = z^{1/2} q^{(j-1)/2}.$$

This proves that indeed $LU = M$. Therefore for the determinant of the Lehmer matrix M we obtain the expression

$$\prod_{j=1}^{n} \frac{\lambda(j)}{\lambda(j-1)} = \frac{\lambda(n)}{\lambda(0)} = \sum_{0 \le k \le n/2} \begin{bmatrix} n-k \\ k \end{bmatrix} (-1)^k q^{k(k-1)} z^k.$$

This is the result posted on August 21, 2018[1] by Ekhad and Zeilberger [5]. Of course, taking the limit $n \to \infty$, leads to the old result by Lehmer for the determinant of the infinite matrix:

$$\lim_{n \to \infty} \det(M(n)) = \sum_{k \ge 0} \frac{(-1)^k q^{k(k-1)} z^k}{(q;q)_k}.$$

Remarks.

1. For $q = 1$, Lehmer's determinant plays a role when enumerating lattice paths (Dyck paths) of bounded height, or planar trees of bounded height, see [4, 7, 6].

2. Recursions as in (1) have been studied in [2, 3, 9] and are linked to so-called Schur polynomials [11].

---

[1] Today.

References

[1] George E. Andrews. *The theory of partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[2] George E. Andrews. Fibonacci numbers and the Rogers-Ramanujan identities. *Fibonacci Quart.*, 42(1):3–19, 2004.

[3] Johann Cigler. Some algebraic aspects of Morse code sequences. *Discrete Math. Theor. Comput. Sci.*, 6(1):55–68, 2003.

[4] N. G. de Bruijn, D. E. Knuth, and S. O. Rice. The average height of planted plane trees. In *Graph theory and computing*, pages 15–22. Academic Press, New York, 1972.

[5] S. B. Ekhad and D. Zeilberger. D. H. Lehmer's Tridiagonal determinant: An Etude in (Andrews-Inspired) Experimental Mathematics. *arXiv:1808.06730v1*, posted August 21, 2018.

[6] Benjamin Hackl, Clemens Heuberger, Helmut Prodinger, and Stephan Wagner. Analysis of bidirectional ballot sequences and random walks ending in their maximum. *Ann. Comb.*, 20(4):775–797, 2016.

[7] Donald E. Knuth. *Selected papers on analysis of algorithms*, volume 102 of *CSLI Lecture Notes*. CSLI Publications, Stanford, CA, 2000.

[8] D. H. Lehmer. Combinatorial and cyclotomic properties of certain tridiagonal matrices. *Congressus Numerantium*, X:53–74, 1974.

[9] Peter Paule and Helmut Prodinger. Fountains, histograms, and q-identities. *Discrete Math. Theor. Comput. Sci.*, 6(1):101–106, 2003.

[10] M. Petkovsek, H. S. Wilf, and D. Zeilberger. *"A=B"*. A. K. Peters, 1996.

[11] I. Schur. Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche. *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.*, pages 302–321, 1917. reprinted in: Gesammelte Abhandlungen, Vol. 2.

Helmut Prodinger, Mathematics Department, Stellenbosch University, 7602 Stellenbosch, South Africa.

E-mail address: `hproding@sun.ac.za`