ON THE AVERAGE OSCILLATION OF THE CONTOUR OF MONOTONICALLY LABELLED ORDERED TREES

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Summary

The paper deals with ordered trees the nodes of which are labelled by elements of $\{1,2,\ldots,k\}$ such that any sequence of labels connecting the root with a leaf is weakly monotone. Considering the leaves to be enumerated from left to right an asymptotic result on the average height of the j-th MIN-turn, i.e. the root of the subtree of minimal height with leaves $j,j+1$, for fixed j and large node number n is derived. Combined with a result of [9] on the average height of the j-th leaf, the theorem gives information on the average oscillation of the contour of monotonically labelled ordered trees.

1. Introduction

A large number of recent papers in Discrete Mathematics deal with the problem of the determination of the average shape of rooted tree structures. Compare e.g. [1,2,5,7,11,12,14]. These problems have also proved to be applicable in the analysis of special algorithms (see [8]).

In some of the new work of the authors it has turned out to be interesting to investigate generalized classes of tree structures which are derived from binary trees, t-ary trees, ordered trees etc. by means of a monotone labelling of the nodes of the tree: Consider the nodes labelled by elements of $\{1,2,\ldots,k\}$ in such a way that any sequence of labels connecting the root with a leaf is weakly monotone. In [13] Prodinger and Urbanek have considered the problem of finding asymptotic equivalents to the numbers of such tree structures, in [10] the average height of monotonically labelled binary trees has been investigated and in [9] the average height of the j-th leaf (where leaves are enumerated from left to right) has been established for certain families.

In the present paper these considerations are completed by a more detailed study of the contour of monotonically labelled ordered trees. Our results contain some of the material of Kemp's investigation [8] on the average oscillation of a stack during postorder traversing of a binary tree as a special case.

The mathematical apparatus to handle the problems in question is influenced by some methods and ideas of [9] and seems to give a quite fitting framework of studying some of the already published results in a shorter and probably more legible fashion.

Adopting the notation of Kemp we use the terms MAX-turn and MIN-turn of ordered trees, where the MAX-turns are just the leaves of the tree (i.e. the nodes having no son), which are assumed to be enumerated from left to right by the natural numbers. The j-th MIN-turn is defined as the root of the (uniquely determined) subtree which has exactly the two leaves j and j+1. For example consider the following tree with MAX-turns 1,2,3,4 and MIN-turns 1',2',3':

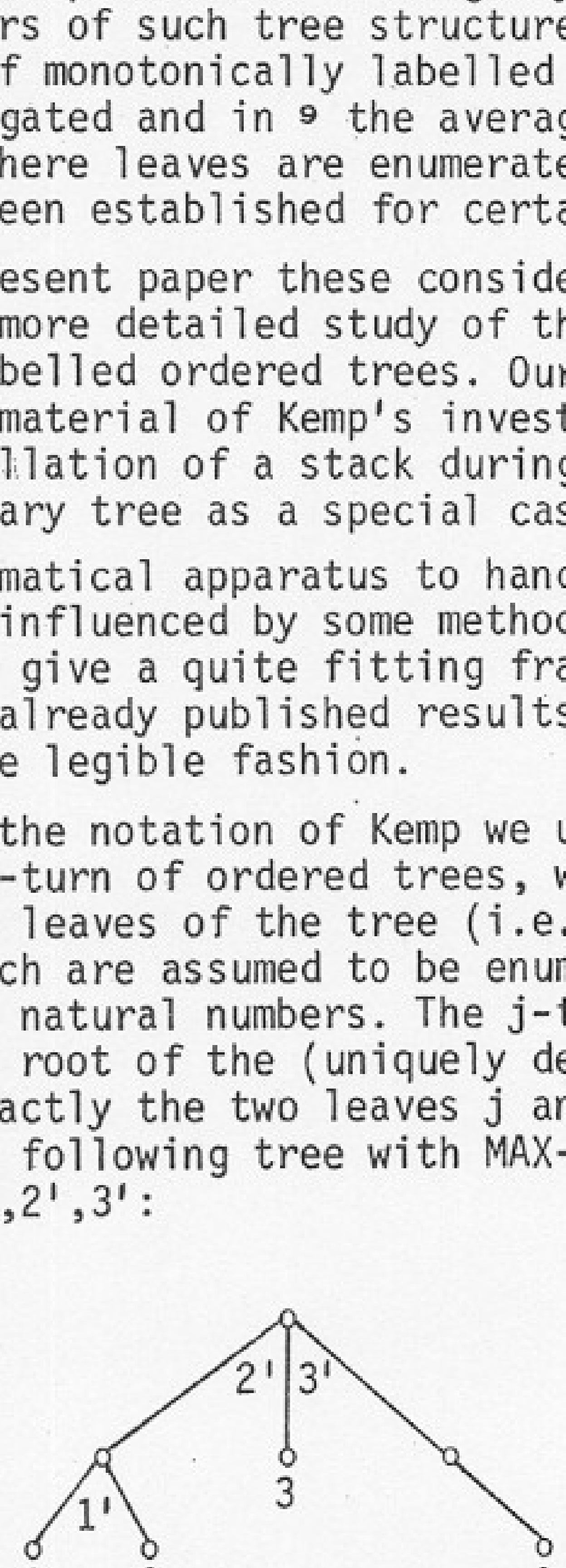

Fig. 1

The level number of a MAX- or MIN-turn is the number of nodes in the chain connecting the root with the turn in question.

In the paper [9] the following result has been established: Let B_k be the family of ordered trees with nodes labelled monotonically of $\{1,2,\ldots,k\}$. The average level number of the j-th leaf (i.e. the j-th MAX-turn) of the trees of B_k with exactly n nodes (where all such trees are regarded equally likely) converges for $n\to\infty$ to a finite limit $\alpha_{MAX}(k,j)$, which can be determined explicitly for small values of k and has the following asymptotic behaviour:

$$\alpha_{MAX}(k,j) \sim C(k).j^{1/2} \quad (j\to\infty) \quad .$$

The constants $C(k)$ can be expressed in terms of some "characteristic quantities" of the family B_k, which have been studied extensively in the paper [13]. This allows to describe the asymptotic behaviour of $C(k)$ as

$$C(k) \sim C.k^{-1/2} \quad (k\to\infty)$$

with a constant C independent from k.

For the following considerations we will use the cited results of [9], but it will turn out to be necessary to study the asymptotic behaviour of $\alpha_{MAX}(k,j)$ more in detail by establishing a result of the type

$$\alpha_{MAX}(k,j) = C_1(k).j^{1/2} + C_2(k) + O(j^{-1/2}) \ (j\to\infty) \ (\P)$$

Furthermore we introduce the average level numbers $\alpha_{MIN}(k,j)$ of the j-th MIN-turn of the trees of B_k with exactly n nodes "for large n" (that is we consider the limit of the concerned numbers for $n\to\infty$), as above in the definition of $\alpha_{MAX}(k,j)$. We will prove that

$$\alpha_{MAX}(k,j) \sim \alpha_{MIN}(k,j) \quad (j\to\infty)$$

and determine the asymptotic behaviour of the difference

$$\alpha_{MAX}(k,j) - \alpha_{MIN}(k,j) \quad (j\to\infty)$$

(which is just the "oscillation of the contour of the trees) by means of the above mentioned result (¶) and a corresponding estimation

$$\alpha_{MIN}(k,j) = C_1(k).j^{1/2} + C_3(k) + O(j^{-1/2}) \quad (j\to\infty)$$

for the MIN-turns.

In the special case k=1 our results yield just the previously cited theorem of Kemp, which in our terminology reads

$$\alpha_{MAX}(1,j) = \frac{8}{\sqrt{2\pi}}.j^{1/2} + 1 + O(j^{-1/2})$$

and

$$\alpha_{MIN}(1,j) = \frac{8}{\sqrt{2\pi}}.j^{1/2} - 1 + O(j^{-1/2}) \ .$$

Remark. For the sake of brevity we will frequently use the symbol "*" as to indicate that the concerned relation is valid if all "*"'s are replaced by "MAX" or all are replaced by "MIN".

2. The average oscillation of ordered trees

Using the suggestive terminology of Flajolet [2] the families B_k of ordered trees with nodes labelled monotonically by elements of $\{1,2,\ldots,k\}$ may be defined by the formal equations

$$\tilde{B}_o = \emptyset \qquad (2.1)$$

$$B_k = \tilde{B}_{k-1} + \textcircled{1} + \overset{\textcircled{1}}{\underset{B_k}{|}} + \overset{\textcircled{1}}{\underset{B_k \quad B_k}{/\ \backslash}} + \ldots$$

where $\tilde{B}_k$ is the family with labels taken from $\{2,\ldots,k+1\}$. The corresponding generating functions $y_k(z)$ fulfill

$$y_o = 0,\ y_k = y_{k-1} + \frac{z}{1-y_k} \quad (k\geq 1) \qquad (2.2)$$

Let $y^*_{k;n,\lambda}$ be the number of trees in B_k with exactly n nodes and λ $*$-turns (for the symbol $*$ compare the remark at the end of the introduction). It is useful to define the following generating functions:

$$y^*_{k,j}(z,u) := \sum_{n\geq 1,\lambda\geq j} y^*_{k;n,\lambda} z^n u^\lambda$$

$$y^*_{k,j}(z) := y^*_{k,j}(z,1) \qquad (2.3)$$

$$y^*_k(z,u) := y^*_{k,0}(z,u)$$

Of course we have

$$y^{MAX}_{k,j}(z,u) = u.y^{MIN}_{k,j-1}(z,u) \quad \text{and}$$

$$y_k(z) = y^{MAX}_{k,1}(z) = y^{MIN}_{k,0}(z). \qquad (2.4)$$

The following recursions hold:

$$y^{MAX}_{k,j}(z) = y^{MAX}_{k-1,j}(z) + z.\delta_{j,1} + \qquad (2.5)$$

$$+ z\sum_{t\geq 1}\sum_{r=0}^{t-1} y_k(z)^{t-1-r}\sum_{i=0}^{j-1}\langle (y_k^{MAX}(z,u))^r, u^i\rangle y^{MAX}_{k,j-i}(z),$$

$$y^{MIN}_{k,j}(z) = y^{MIN}_{k-1,j}(z) +$$

$$+ z\sum_{t\geq 1}\sum_{r=0}^{t-1} y_k(z)^{t-1-r}\sum_{i=0}^{j-1-r}\langle (y_k^{MIN}(z,u))^r, u^i\rangle y^{MIN}_{k,j-1-r}(z)$$

$$+ z\sum_{t\geq 2}\sum_{r=1}^{t-1} y_k(z)^{t-r}\langle (y_k^{MIN}(z,u))^r, u^{j-r}\rangle, \quad (j\geq 1)$$

where $\langle f(u),u^i\rangle$ denotes the coefficient of u^i in the (formal) power series $f(u)$.

The first recursion is proved in [9] (4.6); for the second recursion observe that a tree in B_k may either have a root labelled by an element of $\{2,\ldots,k\}$, which constitutes the first term, or it starts with a root labelled by 1. In the latter case there are 2 possibilities for the position of the MIN-turn: either the j-th MIN-turn is situated within the r+1-st subtree of the root, which yields the second term ($0\leq r\leq t-1$), or the j-th MIN-turn lies in the root itself, which constitutes the third term.

Next we consider the generating functions $v^{*,[h]}_{k,j}(z)$ of trees in B_k with at least j $*$-turns and with a level number of the j-th $*$-turn which is less of equal to h. By a similar argument as above we have

$$v^{MAX,[h+1]}_{k,j}(z) = v^{MAX,[h+1]}_{k-1,j}(z) + z.\delta_{j,1} +$$

$$+ z\sum_{t\geq 1}\sum_{r=0}^{t-1} y_k(z)^{t-1-r}\sum_{i=0}^{j-1}\langle (y_k^{MAX}(z,u))^r, u^i\rangle v^{MAX,[h]}_{k,j-i}(z),$$

$$v^{MIN,[h+1]}_{k,j}(z) = v^{MIN,[h+1]}_{k-1,j}(z) + \qquad (2.6)$$

$$+ z\sum_{t\geq 1}\sum_{r=0}^{t-1} y_k(z)^{t-1-r}\sum_{i=0}^{j-1-r}\langle (y_k^{MIN}(z,u))^r, u^i\rangle v^{MIN,[h]}_{k,j-i-r}(z)$$

$$+ z\sum_{t\geq 2}\sum_{r=1}^{t-1} y_k(z)^{t-r}\langle (y_k^{MIN}(z,u))^r, u^{j-r}\rangle \qquad (j\geq 1)$$

with

$$v^{MAX,[0]}_{k,j}(z) = v^{MIN,[0]}_{k,j}(z) = 0.$$

Let

$$H^*_{k,j}(z) := \sum_{h\geq 0}(y^*_{k,j}(z) - v^{*,[h]}_{k,j}(z));\quad H^*_k(z,u) := \sum_{j\geq 1} H^*_{k,j}(z)u^j. \qquad (2.7)$$

The coefficient of z^n in these power series is just the sum of level numbers of the j-th $*$-turn.

In [9] (4.12) the following recursion is established:

$$H_k^{MAX}(z,u).\frac{\Delta^{MAX}_{k-1}(z,u)+z(1-u)}{\Delta^{MAX}_k(z,u)} = \qquad (2.8)$$

$$= H^{MAX}_{k-1}(z,u) + \frac{u}{1-u}(\Delta^{MAX}_k(z,u) - \Delta^{MAX}_{k-1}(z,u))$$

where

$$\Delta^{MAX}_i(z,u) := y_i(z) - y^{MAX}_i(z,u). \qquad (2.9)$$

Subtracting recursions (2.5) and (2.6) for the MIN-turns and summing up we obtain

$$H^{MIN}_k(z,u).\left[1 - \frac{z}{1-uy^{MIN}_k(z,u)}\cdot\frac{1}{1-y_k(z)}\right] = \qquad (2.10)$$

$$= H^{MIN}_{k-1}(z,u) + \frac{u}{1-u}(\Delta^{MIN}_k(z,u) - \Delta^{MIN}_{k-1}(z,u))$$

where

$$\Delta^{MIN}_i(z,u) := y_i(z) - y^{MIN}_i(z,u). \qquad (2.11)$$

Observing

$$uy^{MIN}_k(z,u) = y^{MAX}_k(z,u),$$

$$u\Delta^{MIN}_i(z,u) = \Delta^{MAX}_i(z,u) + (u-1).y_i(z)$$

and

$$1 - \frac{z}{1-y^{MAX}_k(z,u)}\cdot\frac{1}{1-y_k(z)} = \frac{\Delta^{MAX}_{k-1}(z,u) + z(1-u)}{\Delta^{MAX}_k(z,u)}$$

(compare [9] (7.11f)) recursion (2.10) is transformed into

$$H^{MIN}_k.\frac{\Delta^{MAX}_{k-1} + z(1-u)}{\Delta^{MAX}_k(z,u)} = \qquad (2.12)$$

$$= H^{MIN}_{k-1} + \frac{1}{1-u}(\Delta^{MAX}_k - \Delta^{MAX}_{k-1}) - (y_k(z) - y_{k-1}(z)).$$

So we have

$$H^{MIN}_k = \frac{1}{u}.H^{MAX}_k - G_k \qquad (2.13)$$

where the functions $G_k(z,u)$ fulfill

$$G_o = 0$$

$$G_k \cdot \frac{\Delta_{k-1}^{MAX} + z(1-u)}{\Delta_k^{MAX}} = G_{k-1} + (y_k(z) - y_{k-1}(z)) \tag{2.14}$$

and therefore

$$G_k(z,u) = \sum_{i=1}^{k} (y_i(z)-y_{i-1}(z)) \prod_{j=i}^{k} \frac{\Delta_j^{MAX}}{\Delta_{j-1}^{MAX} + z(1-u)} . \tag{2.15}$$

In the paper [13] it has been shown that the (algebraic) singularities q_k nearest to the origin of the functions $y_k(z)$ obey the following recursion:

$$q_1 = 1/4$$
$$q_{i+1}^{-1} = q_i^{-1} + q_i + 2 \tag{2.16}$$

and the functions $y_k(z)$ behave like

$$y_k(z) = y_k(q_k) - a_k\cdot(q_k-z)^{1/2} + O(q_k-z), \ (z\to q_k-) . \tag{2.17}$$

From this it is not difficult to conclude that the behaviour of the functions $H^*_{k,j}$ is of the following type:

$$H^*_{k,j}(z) = H^*_{k,j}(q_k) - a^*_{k,j}\cdot(q_k-z)^{1/2} + O(q_k-z) . \tag{2.18}$$

The asymptotic behaviour of the coefficients follows now by a theorem of Darboux (see e.g.[4,11]) to be

$$\langle y_k,z^n\rangle \sim \frac{a_k}{2}\cdot(\frac{q_k}{\pi})^{1/2}.q_k^{-n}.n^{-3/2} ,$$
$$\langle H^*_{k,j},z^n\rangle \sim \frac{a^*_{k,j}}{2}\cdot(\frac{q_k}{\pi})^{1/2}.q_k^{-n}.n^{-3/2} \quad (n\to\infty). \tag{2.19}$$

The desired average level numbers of the j-th $*$-turn "for large n" (compare the comments in the introduction) are therefore given by

$$a_*(k,j) = \lim_{n\to\infty} \frac{\langle H^*_{k,j},z^n\rangle}{\langle y_k,z^n\rangle} = \frac{a^*_{k,j}}{a_k} . \tag{2.20}$$

We introduce the generating functions

$$A_k^*(u) := \sum_{j\geq 1} \alpha_*(k,j).u^j . \tag{2.21}$$

Relations (2.20) may be rewritten now in shorter form:

$$a_k.A_k^*(u) = \lim_{z\to q_k-} \frac{H_k^*(q_k,u) - H_k^*(z,u)}{(q_k-z)^{1/2}} . \tag{2.22}$$

By relation (2.13) we have

$$A_k^{MIN}(u) = \frac{1}{u} . A_k^{MAX}(u) - B_k(u) \tag{2.23}$$

where $B_k(u)$ is defined by the relation

$$a_k.B_k(u) = \lim_{z\to q_k-} \frac{G_k(q_k,u) - G_k(z,u)}{(q_k-z)^{1/2}} . \tag{2.24}$$

In the following we will expand the functions $A_k(u)$ as

$$A_k^{MAX}(u) = \frac{\alpha_k}{(1-u)^{3/2}} + \frac{\beta_k}{1-u} + O((1-u)^{-1/2}) , \tag{2.25}$$

$$B_k(u) = \frac{\gamma_k}{1-u} + O((1-u)^{-1/2}), \quad (u\to 1-).$$

In [9] (4.17) $A_k^{MAX}(u)$ is established to be

$$A_k^{MAX}(u) = \frac{H_{k-1}^{MAX}(q_k,u)}{\Delta_{k-1}^{MAX}(q_k,u) + q_k(1-u)} + \frac{u}{1-u} \cdot \frac{2\Delta_k^{MAX}(q_k,u) - \Delta_{k-1}^{MAX}(q_k,u)}{\Delta_{k-1}^{MAX}(q_k,u) + q_k(1-u)} \tag{2.26}$$

and (see [9] (4.22))

$$\alpha_k = 2.q_k^{-1/4}.(q_k + d_{k-1})^{-1/2} , \tag{2.27}$$

where $(i\leq k-1)$

$$d_i = \left.\frac{\partial y_i^{MAX}(z,u)}{\partial u}\right|_{u=1,z=q_k} = q_k\cdot\sum_{r=1}^{i}\prod_{j=1}^{r}\frac{1}{1-q_j} =$$

$$= q_k\cdot\sum_{r=1}^{i}[q]_r , \text{ with } [q]_r := \prod_{j=1}^{r}\frac{1}{1-q_j} . \tag{2.28}$$

(Compare [9](4.25)).

With the first identity for d_i it is obvious that

$$\Delta_{k-1}^{MAX}(q_k,u) = d_{k-1}\cdot(1-u) + O((1-u)^2) . \tag{2.29}$$

Therefore we have by (2.26)

$$\beta_k = \frac{H_{k-1}^{MAX}(q_k,1)}{d_{k-1} + q_k} + \omega_k , \tag{2.30}$$

where ω_k is the coefficient of $(1-u)^0$ in the expansion of

$$\frac{2\Delta_k^{MAX}(q_k,u) - \Delta_{k-1}^{MAX}(q_k,u)}{\Delta_{k-1}^{MAX}(q_k,u) + q_k(1-u)} . \tag{2.31}$$

In [9](4.21) it is shown that

$$\Delta_k^{MAX}(q_k,u) = q_k^{1/4}.(q_k + d_{k-1})^{1/2}.(1-u)^{1/2} + C_k\cdot(1-u) + O((1-u)^{3/2}). \tag{2.32}$$

The constant C_k can be determined in the following way: By the definition of Δ_k^{MAX} and formula (4.5) of [9] we have

$$\Delta_k^{MAX}(q_k,u) = y_k(q_k) - y_k^{MAX}(q_k,u) =$$
$$= y_k(q_k) - \frac{1}{2} + \frac{1}{2}q_k(1-u) - \frac{1}{2}y_{k-1}(q_k,u) + \frac{1}{2}\sqrt{(1+q_k(1-u) - y_{k-1}(q_k,u))^2 - 4q_k}$$

and therefore

$$C_k = \frac{q_k + d_{k-1}}{2} . \tag{2.33}$$

As an immediate consequence of (2.29) and (2.32) we get

$$\omega_k = \frac{2C_k - d_{k-1}}{d_{k-1} + q_k} . \qquad (2.34)$$

In order to complete the computation of β_k it remains to determine $H^{MAX}_{k-1}(q_k,1)$: From recursion (2.8) for the functions H^{MAX}_i we get

$$H^{MAX}_{k-1}(q_k,u) =$$

$$= \frac{u}{1-u} \sum_{i=1}^{k-1} (\Delta^{MAX}_i(q_k,u) - \Delta^{MAX}_{i-1}(q_k,u)) \prod_{j=i}^{k-1} \frac{\Delta^{MAX}_j(q_k,u)}{\Delta^{MAX}_{j-1}(q_k,u)+q_k(1-u)}$$

and therefore

$$H^{MAX}_{k-1}(q_k,1) = \sum_{i=1}^{k-1} (d_i - d_{i-1}) \prod_{j=i}^{k-1} \frac{d_j}{d_{j-1}+q_k} .$$

By [9] (4.24) we have

$$q_k + d_{i-1} = d_i . (1 - q_{k-i}) \qquad (2.35)$$

and thus

$$\beta_k = \frac{1}{d_{k-1}+q_k} \left[q_k + \sum_{i=1}^{k-1} (d_{k-i} - d_{k-i-1}).[q]_i \right] .$$

By (2.28)

$$d_{k-i} - d_{k-i-1} = q_k \cdot [q]_{k-i} ,$$

so that finally

$$\beta_k = \frac{1 + \sum_{i=1}^{k-1} [q]_{k-i}[q]_i}{1 + \sum_{i=1}^{k-1} [q]_i} . \qquad (2.36)$$

The defining relation for $B_k(u)$, (2.24) and recursion (2.14) for the functions G_k yield

$$B_k(u) = \frac{G_{k-1}(q_k,u)}{\Delta^{MAX}_{k-1}(q_k,u)+q_k(1-u)} + \frac{y_k(q_k)-y_{k-1}(q_k)}{\Delta^{MAX}_{k-1}(q_k,u)+q_k(1-u)} +$$

$$+ \frac{\Delta^{MAX}_k(q_k,u)}{\Delta^{MAX}_{k-1}(q_k,u)+q_k(1-u)} . \qquad (2.37)$$

Now

$$y_k(q_k) - y_{k-1}(q_k) = q_k^{1/2} ,$$

so that we find

$$B_k(u) = \frac{1}{1-u} \left[\frac{G_{k-1}(q_k,1)}{d_{k-1}+q_k} + \frac{q_k^{1/2}}{d_{k-1}+q_k} \right] + O((1-u)^{-1/2}. \qquad (2.38)$$

By (2.15)

$$G_{k-1}(q_k,1) = \sum_{i=1}^{k-1} (y_i(q_k) - y_{i-1}(q_k)) \prod_{j=i}^{k-1} \frac{d_j}{d_{j-1}+q_k}$$

Using again (2.35) and

$$y_i(q_k) - y_{i-1}(q_k) = (q_k \cdot q_{k-i})^{1/2}$$

this is simplified to

$$G_{k-1}(q_k,1) = q_k^{1/2} . \sum_{i=1}^{k-1} q_i^{1/2} . [q]_i . \qquad (2.39)$$

Putting everything together

$$B_k(u) = \frac{\gamma_k}{1-u} + O((1-u)^{-1/2})$$

with

$$\gamma_k = q_k^{-1/2} . \frac{1 + \sum_{i=1}^{k-1} q_i^{1/2}.[q]_i}{1 + \sum_{i=1}^{k-1} [q]_i} . \qquad (2.40)$$

Formulas (2.23) and (2.25) yield

$$A^{MAX}_k(u) = \frac{\alpha_k}{(1-u)^{3/2}} + \frac{\beta_k}{1-u} + O((1-u)^{-1/2}) ,$$

$$A^{MIN}_k(u) = \frac{\alpha_k}{(1-u)^{3/2}} + \frac{\beta_k - \gamma_k}{1-u} + O((1-u)^{-1/2}) \qquad (2.41)$$

and therefore we get by Darboux's theorem cited above

$$\alpha_{MAX}(k,j) = \frac{2}{\sqrt{\pi}} \cdot \alpha_k . j^{1/2} + \beta_k + O(j^{-1/2}) ,$$

$$\alpha_{MIN}(k,j) = \frac{2}{\sqrt{\pi}} \cdot \alpha_k . j^{1/2} + \beta_k - \gamma_k + O(j^{-1/2}) . \qquad (2.42)$$

We summarize our results in the following

<u>THEOREM 1.</u> The average level number of the j-th MAX-(MIN)-turn of an ordered tree with n nodes labelled monotonically by elements of $\{1,2,\ldots,k\}$ has the following asymptotic behaviour "for large n":

$$\alpha_{MAX}(k,j) = C_1(k).j^{1/2} + C_2(k) + O(j^{-1/2}),$$

$$\alpha_{MIN}(k,j) = C_1(k).j^{1/2} + C_3(k) + O(j^{-1/2}), \qquad (j \to \infty)$$

where

$$C_1(k) = \frac{4}{\sqrt{\pi} q_k^{1/2} (1+\sum_{i=1}^{k-1} [q]_i)} ,$$

$$C_2(k) = \frac{1 + \sum_{i=1}^{k-1} [q]_{k-i}[q]_i}{1 + \sum_{i=1}^{k-1} [q]_i}$$

and

$$C_3(k) = C_2(k) - \frac{1}{\sqrt{q_k}} \cdot \frac{1 + \sum_{i=1}^{k-1} q_i^{1/2}[q]_i}{1 + \sum_{i=1}^{k-1} [q]_i}$$

with

$$[q]_i = \prod_{j=1}^{i} \frac{1}{1-q_j} ,$$

and the numbers q_i fulfill the recursion

$$q_{i+1}^{-1} = q_i^{-1} + q_i + 2 , \quad q_1 = 1/4 .$$

This result may be interpreted in the following way: The average difference between the level numbers of a MAX-turn and the consecuting MIN-turn (i.e. the average oscillation of the contour of the tree) is given asymptotically by

$$\alpha_{MAX}(k,j) - \alpha_{MIN}(k,j) = \gamma_k + O(j^{-1/2}),(j\to\infty) \quad (2.43)$$

with γ_k from formula (2.40).

We start with some examples: The instance k=1 is just the case studied in Kemp's paper :
Using the formulas from above we have

$$C_1(1) = \frac{8}{\sqrt{\pi}}, \quad C_2(1) = 1, \quad C_3(1) = -1,$$

and therefore the average oscillation is given by

$$\alpha_{MAX}(1,j) - \alpha_{MIN}(1,j) = 2 + O(j^{-1/2}).$$

In the case k=2, which is the case of monotone Boolean labelling, we get

$$C_1(2) = 4.\sqrt{\frac{15}{14\pi}}, \quad C_2(2) = \frac{25}{21}, \quad C_3(2) = -\frac{25}{42}$$

and for the average oscillation

$$\alpha_{MAX}(2,j) - \alpha_{MIN}(2,j) = \frac{25}{14} + O(j^{-1/2}).$$

The asymptotic behaviour of the constants γ_k can be established as follows.

THEOREM 2. As $k\to\infty$,

$$\gamma_k = \frac{3}{2} + O(\frac{\log k}{k}).$$

Proof.
In [13] it has been shown that the sequence (q_k) fulfills

$$q_k = \frac{1}{2k} + O(\frac{\log k}{k^2}). \quad (2.44)$$

Defining $p_k := q_k^{-1}$ it follows that

$$p_k = 2k + O(\log k).$$

In order to derive one further term of the asymptotic expansion we set

$$p_k = 2k + r_k.$$

Inserting this into the recursion

$$p_{k+1} = p_k + 2 + p_k^{-1},$$

we find

$$p_k = 2k + \frac{1}{2}.\log k + O(1). \quad (2.45)$$

In the next step we treat the terms $[q]_i$:

$$\log [q]_i = \log [q]_{i-1} - \log (1-q_i)$$

and by (2.45)

$$= \log [q]_{i-1} + \frac{1}{2i} - \frac{\log i}{8i^2} + O(\frac{1}{i^2}),$$

so that by a standard summation method

$$\log [q]_i = \frac{1}{2}.\log i + C + \frac{\log i}{8i} + O(\frac{1}{i})$$

where C denotes some constant. Taking exponentials we obtain with some constant D

$$[q]_i = D.i^{1/2} + D.\frac{\log i}{8i^{1/2}} + O(\frac{1}{i^{1/2}}) \quad (2.46)$$

Summing up

$$\sum_{i=1}^{k-1} [q]_i = D.\frac{2}{3}.k^{3/2} + O(k^{1/2}.\log k) \quad (2.47)$$

and

$$\sum_{i=1}^{k-1} q_i^{1/2}.[q]_i = \sum_{i=1}^{k-1} \frac{1}{\sqrt{2i}}(1 - \frac{\log i}{8i} + O(\frac{\log^2 i}{i^2})).$$

$$.(D.i^{1/2} + D.\frac{\log i}{8i^{1/2}} + O(\frac{1}{i^{1/2}}))$$

$$= \frac{D}{\sqrt{2}}.k + O(\log k). \quad (2.48)$$

Inserting these results in formula (2.40) for γ_k, we find

$$\gamma_k = \sqrt{2k}(1 + O(\frac{\log k}{k})).\frac{\frac{D}{\sqrt{2}}.k + O(\log k)}{D.\frac{2}{3}.k^{3/2}+O(k^{1/2}.\log k)}$$

$$= \frac{3}{2} + O(\frac{\log k}{k})$$

and the Theorem is proved.-

The asymptotic behaviour of $C_1(k)$ resp. $C_2(k)$ can be established in a similar way.

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