# ENUMERATION OF DEUTSCH PATHS BY THE ADDING-A-NEW-SLICE METHOD AND APPLICATIONS

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ABSTRACT. A variation of Dyck paths allows for down-steps of arbitrary length, not just one. This is motivated by ideas due to Emeric Deutsch. We use the adding-a-new-slice technique and the kernel method to compute the number of maximal runs of up-step runs of length 1 and a subclass of Deutsch paths satisfying a condition that was stipulated by R. Stanley for Dyck paths.

## 1. INTRODUCTION

Deutsch paths are like Dyck paths, but extra down-steps of the form $(1,-j)$, any $j \geq 2$, are also allowed. They were analyzed recently in [7]. Here, we want to enumerate them in a different manner which is quite versatile when certain parameters of Deutsch paths should be analyzed.

We decompose a Deutsch path into maximal runs of up- resp. down-steps. First, we restrict our attention to the instance when the path ends with down-steps. If a path is closed, this happens anyway (except for the empty path), but for open-ended paths, the last step of the path might be an up-step.

The technique we are using can be found in [1].

We consider two applications: It is well-known that the $n$-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ enumerates Dyck paths of length $2n$. In [8], Stanley lists a variety of other combinatorial interpretations of the Catalan numbers, one of them being the number of Dyck paths from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive $(1,1)$ steps ending on the $x$-axis has odd length. At this point it is interesting to note that there are more subclasses of Dyck paths, also enumerated by Catalan numbers, that are defined via parity restrictions on the length of the returns to the $x$-axis as well (see, e.g., [8]). This restriction of Dyck paths that leads again to Catalan numbers was further investigated in [4]. In this paper, we consider Deutsch paths with the property that each maximal run of down-steps to the $x$-axis starts at an odd level. Unfortunately, in this context, this does not lead to any known/resp. nice numbers.

In the last chapter, we count the number of maximal sequences of up-steps consisting of only one up-step in Deutsch paths.

The paper [5] counted the number of maximal runs of up-steps of length one in Dyck paths. This was greatly extended in [3]; the method of choice in the first incarnation of this paper was indeed the adding-a-new-slice technique combined with the kernel method. We keep our analysis in the instance of Deutsch paths simple and only consider the basic case, leaving more general considerations for later.

## 2. Enumeration of Deutsch paths by the adding-a-new-slice technique

We consider a generating function $F_k(z,u)$, where $k$ is the number of 'mountains' (runs of up-steps, followed by runs of down-steps), $z$ marks the length of the path and $u$ is used to remember the last level reached (coefficient of $u^i$).

The extra down-steps require some preparations. If one want to go down by $h\ge 1$ levels, and do this in $n\ge 1$ steps, it can be done in

$$[z^h]\Big(\frac{z}{1-z}\Big)^n=[z^{h-n}](1-z)^{-n}=\binom{h-1}{h-n}$$

ways. If one wants to keep $n$ variable, using a generating function, we get

$$\sum_{n=1}^{h}\binom{h-1}{h-n}z^n=z(1+z)^{h-1}.$$

And now we have to compute this:

$$\sum_{k>i}z^{k-i}\sum_{0\le j<k}u^jz(1+z)^{k-1-j}=\frac{z^2(1+z)^{i+1}}{(1-z-z^2)(1+z-u)}-\frac{z^2u^{i+1}}{(1-zu)(1+z-u)}.$$

Consequently we have

$$F_{k+1}(z,u)=\frac{z^2(1+z)}{(1-z-z^2)(1+z-u)}F_k(z,1+z)-\frac{z^2u}{(1-zu)(1+z-u)}F_k(z,u).$$

With

$$\Phi(z,u):=\sum_{k\ge0}F_k(z,u)$$

(arbitrary number of mountains) this leads to

$$\Phi(z,u)-1=\frac{z^2(1+z)}{(1-z-z^2)(1+z-u)}\Phi(z,1+z)-\frac{z^2u}{(1-zu)(1+z-u)}\Phi(z,u)$$

or

$$\frac{z(u-u_1)(u-u_2)}{(1-zu)(1+z-u)}\Phi(z,u)=1+\frac{z^2(1+z)}{(1-z-z^2)(1+z-u)}\Phi(z,1+z),$$

with

$$u_{1,2}=\frac{1+z\pm\sqrt{1-2z-3z^2}}{2z}.$$

As it was shown already in [7], we are in the Motzkin world. Now we use arguments from the kernel method [6]. The factor $(u-u_2)$ must also be a factor of the right-hand side, otherwise there would not be power series expansion around $z=0$. This leads to

$$\frac{1+\frac{z^2(1+z)}{(1-z-z^2)(1+z-u)}\Phi(z,1+z)}{u-u_2}=-\frac{1}{1+z-u}.$$

Further simplification leads to

$$\Phi(z,u) = -\frac{(1-zu)}{z(u-u_1)} = \frac{1-zu}{zu_1(1-u/u_1)}.$$

Setting $u=0$ means that the path ends on the $x$-axis:

$$\Phi(z,0) = \frac{1}{zu_1} = \frac{1+z-\sqrt{1-2z-3z^2}}{2z(1+z)} = 1+z^2+z^3+3z^4+6z^5+15z^6+36z^7+\cdots.$$

As was used already in [7], the substitution $z = \frac{v}{1+v+v^2}$ makes everything prettier:

$$\Phi(z,0) = \frac{1+v+v^2}{1+v}.$$

The function

$$\Phi(z,u)\frac{1}{1-zu}$$

describes Deutsch paths that can also end with up-steps. And if one replaces now $u := 1$, we get so-called open Deutsch paths, that can end at any level:

$$\Phi(z,1)\frac{1}{1-z} = \frac{1}{z(u_1-1)} = 1+v+v^2,$$

which also enumerates Motzkin paths. This was explained via a bijection in [7].

## 3. Deutsch paths satisfying a condition by Stanley

As mentioned above, the function

$$\Phi(z,u)\frac{1}{1-zu}$$

describes Deutsch paths that can also end with up-steps. Consequently,

$$\mathscr{G}(z,u) = \Phi(z,u)\frac{zu}{1-zu} = \frac{u}{u_1(1-u/u_1)} = \sum_{k\ge1}\frac{u^k}{u_1^k}$$

is the generating function of paths ('good' paths) ending with an up-step. From this we see that the good paths ending on level $k$ have generating function $1/u_1^k$.

In the spirit of Stanley, we now compute good Deutsch paths, ending on the odd level $2k+1$, and return after that for the first time in a series of down-steps to the $x$-axis:

$$z\sum_{k\ge0}\frac{1}{u_1^{2k}}\cdot z(1+z)^{2k} = \frac{z^2}{1-(1+z)^2/u_1^2}.$$

The final step is to consider an arbitrary sequence of such paths, viz.

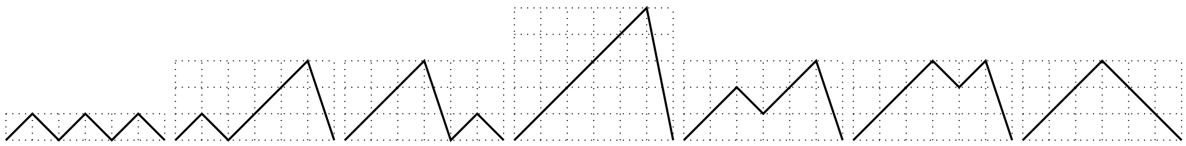
$$\frac{1}{1-\frac{z^2}{1-(1+z)^2/u_1^2}} = \frac{3+z-\sqrt{1-2z-3z^2}}{2(1+z)} = \frac{1+2v+2v^2}{(1+v)^2}.$$

This series is

$$1+z^2+2z^4+2z^5+7z^6+14z^7+37z^8+90z^9+233z^{10}+\cdots,$$

and the coefficients do not bare any significance to Motzkin numbers.

For interest, here are the 7 objects of length 6, satisfying the Stanley-condition that each down-run to the $x$-axis starts at an odd level.



The first and the last are Dyck-paths, and 2 is indeed the Catalan number $C_2$.

## 4. Counting runs of single up-steps

The approach is quite similar to the previous sections; however, we use a third variable, $t$, to count the up-runs of length one.

We have to compute this:

$$\sum_{k\ge i+1} z^{k-i}\sum_{0\le j<k} u^j z(1+z)^{k-1-j} + tz\sum_{0\le j\le i} u^j z(1+z)^{i-j},$$

which leads to

$$F_{k+1}(z,u)=\alpha F_k(z,u)+\beta F_k(z,1+z)$$

with

$$\alpha=\frac{z^2u(-zu+tzu-t)}{(zu-1)(u-1-z)}\quad\text{and}\quad \beta=-\frac{z^2(1+z)(-z-z^2+tz+tz^2-t)}{(u-1-z)(z+z^2-1)}.$$

This leads to

$$\Phi(z,u)-1=\alpha\Phi(z,u)+\beta\Phi(z,1+z)$$

or

$$\frac{z(1+z^2-tz^2)(u-u_1)(u-u_2)}{(1-zu)(1+z-u)}\Phi(z,u)=1+\beta\Phi(z,1+z),$$

The two roots are now

$$u_{1,2}=\frac{-tz^2+z+1+z^2\pm\sqrt{t^2z^4+2tz^3-2tz^2+2tz^4-z^2-2z-2z^3+1-3z^4}}{2z(1+z^2-tz^2)}.$$

Simplification, after dividing out the factor $u-u_2$ from the equation, leads to

$$\Phi(z,u)=\frac{(1-zu)}{z(1+z^2-tz^2)(u_1-u)}.$$

This time, we confine ourselves to the instance $u=0$, i.e., Deutsch paths returning to the $x$-axis. We get

$$\Phi(z,0)=\frac{1}{z(1+z^2-tz^2)u_1}$$

$$= \frac{-tz^2+z+1+z^2-\sqrt{t^2z^4+2tz^3-2tz^2+2tz^4-z^2-2z-2z^3+1-3z^4}}{2z(1+z)(z^2-tz^2+1)}$$
$$= 1+tz^2+z^3+(t^2+2)z^4+(3+3t)z^5+(7+7t+t^3)z^6+(17+13t+6t^2)z^7+\cdots.$$

Once one has this generating function, one can state many results as a corollary. We will only provide one such result, namely, compute the average of the parameter labelled by the variable $t$. So, we differentiate $\Phi(z,0)$ w.r.t. $t$ and then set $t:=1$. This leads to

$$\frac{v^2}{(1-v)(1+v)^2(1+v+v^2)}.$$

One could even read off the coefficients from this, but this would we a sum, so we refrain from doing this. However, we are interested in asymptotics. We will use singularity analysis, as is now customary, see [2].

The relevant singularity is at $z=\frac{1}{3}$, and we find, as $z\to\frac{1}{3}$,

$$v\sim 1-\sqrt{3}\sqrt{1-3z}.$$

Furthermore,

$$\frac{v^2}{(1-v)(1+v)^2(1+v+v^2)}\sim\frac{\sqrt{3}}{36\sqrt{1-3z}}.$$

Therefore

$$[z^n]\frac{v^2}{(1-v)(1+v)^2(1+v+v^2)}\sim[z^n]\frac{\sqrt{3}}{36\sqrt{1-3z}}\sim\frac{\sqrt{3}}{36}3^n\frac{1}{\sqrt{\pi n}}.$$

This needs to be divided by the total number of such paths, viz.

$$[z^n]\frac{1+v+v^2}{1+v}\sim[z^n]\Big(\frac{3}{2}-\frac{3\sqrt{3}}{4}\sqrt{1-3z}\Big)\sim\frac{3\sqrt{3}}{8}3^n\frac{1}{\sqrt{\pi n^{3/2}}}.$$

The quotient is

$$\sim\frac{2n}{27}=0.074n.$$

So, a Deutsch path of length $n$ has about $0.074n$ up-runs of length 1. Many such results could be derived with some patience and a computer.

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