

# A CURIOUS IDENTITY PROVED BY CAUCHY'S INTEGRAL FORMULA

HELMUT PRODINGER<sup>†</sup>

Simons [2] has proved the identity

$$\sum_{r=0}^q \frac{(-1)^{q+r} (q+r)! (1+x)^r}{(q-r)! r!^2} = \sum_{r=0}^q \frac{(q+r)! x^r}{(q-r)! r!^2}; \quad (1)$$

Chapman [1] gave a nice and short proof of it. In this note, I want to give another attractive proof. It uses Cauchy's integral formula to pull out coefficients of generating functions.

We divide (1) by  $q!$  and prove the equivalent version

$$S = \sum_{r=0}^q \binom{q}{r} \binom{q+r}{r} (-1)^{q+r} (1+x)^r = \sum_{r=0}^q \binom{q}{r} \binom{q+r}{r} x^r.$$

We start with the righthand-side:

$$\begin{aligned} S &= [t^q] \sum_{i \geq 0} \binom{q}{i} t^i \cdot \sum_{i \geq 0} \binom{q+i}{i} (tx)^i \\ &= [t^q] (1+t)^q \cdot (1-tx)^{-q-1} \\ &= \frac{1}{2\pi i} \oint \frac{dt}{t^{q+1}} (1+t)^q \cdot (1-tx)^{-q-1}. \end{aligned}$$

Now we substitute  $t = u/(1-u)$ , so that  $dt = du/(1-u)^2$  and obtain

$$\begin{aligned} S &= \frac{1}{2\pi i} \oint \frac{du}{(1-u)^2} \frac{(1-u)^{q+1}}{u^{q+1}} (1-u)^{-q} \cdot \left( \frac{1-u(1+x)}{1-u} \right)^{-q-1} \\ &= [u^q] (1-u)^q (1-u(1+x))^{-q-1} \\ &= \sum_{r=0}^q \binom{-q-1}{r} (-1)^r (1+x)^r \binom{q}{q-r} (-1)^{q-r} \\ &= \sum_{r=0}^q \binom{q+r}{r} \binom{q}{r} (1+x)^r (-1)^{q-r}, \end{aligned}$$

which is the lefthand-side.

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## REFERENCES

- [1] R. Chapman. A curious identity revisited. *The Mathematical Gazette*, 87:139–141, 2003.
- [2] S. Simons. A curious identity. *The Mathematical Gazette*, 85:296–298, 2001.

H. PRODINGER, THE JOHN KNOPFMACHER CENTRE FOR APPLICABLE ANALYSIS AND NUMBER THEORY, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, P. O. WITS, 2050 JOHANNESBURG, SOUTH AFRICA

*E-mail address:* `helmut@maths.wits.ac.za`