89.44 A curious identity proved by Cauchy's integral formula*

Simons [1] has proved the identity

$$\sum_{r=0}^{q} \frac{(-1)^{q+r}(q+r)!\,(1+x)^r}{(q-r)!\,(r!)^2} = \sum_{r=0}^{q} \frac{(q+r)!\,x^r}{(q-r)!\,(r!)^2} \qquad (1)$$

Chapman [2] gave a nice and short proof of it. In this note, I want to give another attractive proof. It uses Cauchy's integral formula to pull out coefficients of generating functions.

If $f(z) = \sum_{n \geqslant 0} a_n z^n$, then we write $[z^n]f(z) = a_n$. Further, this coefficient can be recovered by *Cauchy's integral formula*: $a_n = \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}} f(z)$; this integral is a contour integral that encircles the origin once in a positive direction. (One could avoid this language be speaking about *residues of formal Laurent series* instead.) The interested reader can find some background in [3].

We prove the equivalent version

$$S = \sum_{r=0}^{q} \binom{q}{r}\binom{q+r}{r}(-1)^{q+r}(1+x)^r = \sum_{r=0}^{q} \binom{q}{r}\binom{q+r}{r}x^r.$$

We start with the right-hand side:

$$\begin{aligned} S &= [t^q]\left(\sum_{i \geqslant 0}\binom{q}{i}t^i \times \sum_{i \geqslant 0}\binom{q+i}{i}(tx)^i\right) \\ &= [t^q](1+t)^q(1-tx)^{-q-1} \\ &= \frac{1}{2\pi i}\oint \frac{dt}{t^{q+1}}(1+t)^q(1-tx)^{-q-1}. \end{aligned}$$

Now we substitute $t = u/(1-u)$, so that $dt = du/(1-u)^2$ and obtain

$$\begin{aligned} S &= \frac{1}{2\pi i}\oint \frac{du}{(1-u)^2}\;\frac{(1-u)^{q+1}}{u^{q+1}}\;(1-u)^{-q}.\left(\frac{1-u(1+x)}{1-u}\right)^{-q-1} \\ &= \{[u^q](1-u)^q(1-u(1+x))^{-q-1}\} \\ &= \sum_{r=0}^{q}\binom{-q-1}{r}(-1)^r(1+x)^r\binom{q}{q-r}(-1)^{q-r} \\ &= \sum_{r=0}^{q}\binom{q+r}{r}\binom{q}{r}(1+x)^r(-1)^{q-r}, \end{aligned}$$

which is the left-hand side.

* Supported by NRF Grant 2053748

References

1. S. Simons, A curious identity, *Math. Gaz.* **85** (July 2001) pp. 296-298.
2. R. Chapman, A curious identity revisited, *Math. Gaz.* **87** (March 2003) pp. 139-141.
3. I. Goulden and D. Jackson, Combinatorial enumeration, John Wiley (1983).

HELMUT PRODINGER

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits 2050 Johannesburg, South Africa

e-mail: *helmut@maths.wits.ac.za*