A NEW RECURSION FOR BRESSOUD'S POLYNOMIALS

HELMUT PRODINGER

ABSTRACT. A new recursion in only one variable allows very simple verifications of Bressoud's polynomial identities, which lead to the Rogers-Ramanujan identities. This approach might be compared with an earlier approach due to Chapman. Applying the q-Vandermonde convolution, as suggested by Cigler, makes the computations particularly simple and elementary. The same treatment is also applied to the Santos polynomials.

1. Bressoud's polynomial identities

Let

$$A_n = \sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}, \qquad B_n = \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}} \begin{bmatrix} 2n \\ n-2j \end{bmatrix},$$

and

$$C_n = \sum_{k=0}^{n} q^{k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}, \qquad D_n = \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} 2n+1 \\ n+1-2j \end{bmatrix}.$$

Here, $\begin{bmatrix} n \\ k \end{bmatrix}$ are q-binomial coefficients [2].

Bressoud [3] proved that $A_n = B_n$ and $C_n = D_n$ and that taking the limit $n \to \infty$ leads to the celebrated Rogers-Ramanujan identities. Since it is well documented in the literature how to take this limit we will not repeat this here and concentrate on the polynomial identities.

Chapman [4] simplified Bressoud's arguments, and his approach was also used in [2]. A different simple proof was provided by Cigler [5].

Chapman's method consists in showing that both sides of the identity satisfy the same recursion. This recursion is, however, in two variables. Also, auxiliary sequences needed to be introduced.

Here, we use a different recursion that depends only on one variable, and requires no auxiliary sequences.

It is easy to check that the first two values of the sequences also coincide, so that the sequences themselves coincide.

There are other approaches to deal with Bressoud's polynomials and extensions, like [6]. Here, we try to make everything as simple and elementary as possible.

In a final section, we apply the same machinery to the so-called Santos polynomials [1]. They belong to another pair of Rogers-Ramanujan type identities.

2. The first identity

It is a routine computation that

$$\begin{bmatrix}n\\k\end{bmatrix}-(1+q-q^n)\begin{bmatrix}n-1\\k\end{bmatrix}-q^{2n-2k}\begin{bmatrix}n-1\\k-1\end{bmatrix}+q(1-q^{n-1})\begin{bmatrix}n-2\\k\end{bmatrix}=0.$$

Multiplying this by q^{k^2} and summing leads to the recursion

$$A_n-(1+q-q^n+q^{2n-1})A_{n-1}+q(1-q^{n-1})A_{n-2}=0.$$

It is more of a challenge to show the recursion

$$B_n-(1+q-q^n+q^{2n-1})B_{n-1}+q(1-q^{n-1})B_{n-2}=0,$$

which we will do now.

Cigler [5] used the q-Vandermonde convolution to expand

$$\begin{bmatrix}2n\\n-2j\end{bmatrix}=\sum_k q^{(k-j)(k+j)}\begin{bmatrix}n\\k-j\end{bmatrix}\begin{bmatrix}n\\k+j\end{bmatrix}$$

and consequently

$$B_n=\sum_k q^{k^2}f(n,k)$$

with

$$f(n,k):=\sum_j(-1)^jq^{\frac{j(3j-1)}{2}}\begin{bmatrix}n\\k-j\end{bmatrix}\begin{bmatrix}n\\k+j\end{bmatrix}.$$

One can actually compute $f(n,k)=\begin{bmatrix}n\\k\end{bmatrix}$, but it is not required to know that. We only need simple recursions for $f(n,k)$ that appear already in [5] but are repeated here for completeness. For that, only the recursions for the q-binomial coefficients are needed.

We start with

$$\begin{aligned}&\sum_j(-1)^jq^{\frac{j(3j-1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n\\k+j\end{bmatrix}\\&\quad=\sum_j(-1)^jq^{\frac{j(3j-1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n-1\\k+j\end{bmatrix}+q^{n-k}\sum_j(-1)^jq^{\frac{3j(j-1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n-1\\k+j-1\end{bmatrix}\\&\quad=f(n-1,k),\end{aligned}$$

since the last sum, on the substitution $j\to -j+1$, turns into its own negative. Similarly,

$$\begin{aligned}&\sum_j(-1)^jq^{\frac{j(3j+1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n\\k+j+1\end{bmatrix}\\&\quad=q^{k+1}\sum_j(-1)^jq^{\frac{3j(j+1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n-1\\k+j+1\end{bmatrix}+\sum_j(-1)^jq^{\frac{j(3j+1)}{2}}\begin{bmatrix}n-1\\k-j\end{bmatrix}\begin{bmatrix}n-1\\k+j\end{bmatrix}\\&\quad=f(n-1,k),\end{aligned}$$

by the same reasoning. Therefore

$$\begin{aligned} f(n,k) &= \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k+j \end{bmatrix} \left(\begin{bmatrix} n-1 \\ k-j \end{bmatrix} + q^{n-k+j} \begin{bmatrix} n-1 \\ k-j-1 \end{bmatrix} \right) \\ &= f(n-1,k) + q^{n-k} \sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ k+j \end{bmatrix} \begin{bmatrix} n-1 \\ k-j-1 \end{bmatrix} \\ &= f(n-1,k) + q^{n-k} f(n-1,k-1). \end{aligned}$$

A very similar computation leads to

$$\begin{aligned} f(n,k) &= \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \left(q^{k+j} \begin{bmatrix} n-1 \\ k+j \end{bmatrix} + \begin{bmatrix} n-1 \\ k+j-1 \end{bmatrix} \right) \\ &= q^k \sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n-1 \\ k+j \end{bmatrix} + \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n-1 \\ k+j-1 \end{bmatrix} \\ &= q^k \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k+j \end{bmatrix} \begin{bmatrix} n-1 \\ k-j \end{bmatrix} + \sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ k+j \end{bmatrix} \begin{bmatrix} n-1 \\ k-j-1 \end{bmatrix} \\ &= q^k f(n-1,k) + f(n-1,k-1). \end{aligned}$$

Now we can prove that the sequence B_n satisfies the recursion, which means that

$$\begin{aligned} \sum_k q^{k^2} f(n,k) - (1+q-q^n) \sum_k q^{k^2} f(n-1,k) - q^{2n} \sum_k q^{k^2-2k} f(n-1,k-1) \\ + q(1-q^{n-1}) \sum_k q^{k^2} f(n-2,k) = 0. \end{aligned}$$

We claim that even

$$f(n,k) - (1+q-q^n) f(n-1,k) - q^{2n-2k} f(n-1,k-1) + q(1-q^{n-1}) f(n-2,k) = 0.$$

Using the recursion, this is equivalent to

$$\begin{aligned} & q^{n-k} f(n-1,k-1) - (q-q^n) f(n-1,k) - q^{2n-2k} f(n-1,k-1) + q(1-q^{n-1}) f(n-2,k) \\ &= (q^{n-k} - q^{2n-2k}) f(n-1,k-1) - (q-q^n)(f(n-1,k) - f(n-2,k)) \\ &= (q^{n-k} - q^{2n-2k}) f(n-1,k-1) - (q-q^n) q^{n-k-1} f(n-2,k-1) = 0. \end{aligned}$$

This may be further reduced to

$$\begin{aligned} & (1-q^{n-k}) f(n-1,k-1) - (1-q^{n-1}) f(n-2,k-1) \\ &\quad = f(n-1,k-1) - f(n-2,k-1) - q^{n-k}(f(n-1,k-1) - q^{k-1} f(n-2,k-1)) \\ &\quad = q^{n-k} f(n-2,k-2) - q^{n-k} f(n-2,k-2) = 0, \end{aligned}$$

which is now obvious.

3. The second identity

From

$$\begin{bmatrix} n \\ k \end{bmatrix} - (1+q-q^n)\begin{bmatrix} n-1 \\ k \end{bmatrix} - q^{2n-2k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q(1-q^{n-1})\begin{bmatrix} n-2 \\ k \end{bmatrix} = 0,$$

multiplying this by q^{k^2+k} and summing we are led to the recursion

$$C_n - (1+q-q^n+q^{2n})C_{n-1} + q(1-q^{n-1})C_{n-2} = 0.$$

Now we will deduce the recursion

$$D_n - (1+q-q^n+q^{2n})D_{n-1} + q(1-q^{n-1})D_{n-2} = 0$$

as well.

The q-Vandermonde formula leads to

$$\begin{bmatrix} 2n+1 \\ n-2j \end{bmatrix} = \sum_k q^{k^2+k-j^2+j}\begin{bmatrix} n+1 \\ k+1-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix},$$

and therefore

$$\begin{aligned} D_n &= \sum_j (-1)^j q^{\frac{j(5j-3)}{2}} \sum_k q^{k^2+k-j^2+j}\begin{bmatrix} n+1 \\ k+1-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} \\ &= \sum_k q^{k^2+k} \sum_j (-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n+1 \\ k+1-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix} \\ &= \sum_k q^{k^2+k} f(n,k). \end{aligned}$$

From

$$f(n,k) - (1+q-q^n)f(n-1,k) - q^{2n-2k}f(n-1,k-1) + q(1-q^{n-1})f(n-2,k) = 0$$

we get, upon multiplication with q^{k^2+k} and summing

$$D_n - (1+q-q^n)D_{n-1} - q^{2n}D_{n-1} + q(1-q^{n-1})D_{n-2} = 0,$$

as claimed.

4. Santos polynomials

The Santos polynomials are defined as

$$S_n := \sum_{0\le 2k\le n} q^{2k^2}\begin{bmatrix} n \\ 2k \end{bmatrix}.$$

We start from

$$\begin{bmatrix} n+2 \\ 2k \end{bmatrix} - (1+q)\begin{bmatrix} n+1 \\ 2k \end{bmatrix} + q\begin{bmatrix} n \\ 2k \end{bmatrix} - q^{2n+4-4k}\begin{bmatrix} n \\ 2k-2 \end{bmatrix}.$$

Multiplying this by q^{2k^2} and summing, we find the recursion

$$S_{n+2}-(1+q)S_{n+1}+(q-q^{2n+2})S_n=0.$$

(Originally, it was found using Zeilberger's q-EKHAD algorithm.)

The alternative form for the Santos polynomials, as to be shown, is

$$\overline{S}_n=\sum_j q^{4j^2-j}\begin{bmatrix} n \\ \lfloor\frac{n+1}{2}\rfloor-2j\end{bmatrix}_{q^2}.$$

The q-Vandermonde formula leads to

$$\begin{bmatrix} n \\ \lfloor\frac{n+1}{2}\rfloor-2j\end{bmatrix}_{q^2}=\sum_k \begin{bmatrix} \lceil\frac{n}{2}\rceil \\ k+j\end{bmatrix}_{q^2}\begin{bmatrix} \lfloor\frac{n}{2}\rfloor \\ k-j\end{bmatrix}_{q^2}q^{2k^2-2j^2}.$$

Therefore

$$\overline{S}_n=\sum_k q^{2k^2}g(n,j)$$

with

$$g(n,j)=\sum_j q^{2j^2-j}\begin{bmatrix} \lceil\frac{n}{2}\rceil \\ k+j\end{bmatrix}_{q^2}\begin{bmatrix} \lfloor\frac{n}{2}\rfloor \\ k-j\end{bmatrix}_{q^2}.$$

Using only the recursions for the q-binomial coefficients, a tedious computation leads to

$$g(n+2,j)-(1+q)g(n+1,j)+q\,g(n,j)-q^{2n+2-2j}g(n,j-1).$$

Multiplying this by q^{2k^2} and summing, we find the recursion

$$\overline{S}_{n+2}-(1+q)\overline{S}_{n+1}+(q-q^{2n+2})\overline{S}_n=0.$$

Since the recursion for $g(n,j)$ defines, together with some initial conditions, the sequence uniquely, this also shows that $g(n,j)=\left[\begin{smallmatrix} n \\ 2j\end{smallmatrix}\right]$.

There is a second family of Santos polynomials, defined by

$$T_n:=\sum_{0\le 2k+1\le n} q^{2k^2+2k}\begin{bmatrix} n \\ 2k+1\end{bmatrix}.$$

We start from

$$\begin{bmatrix} n+2 \\ 2k+1\end{bmatrix}-(1+q)\begin{bmatrix} n+1 \\ 2k+1\end{bmatrix}+q\begin{bmatrix} n \\ 2k+1\end{bmatrix}-q^{2n+2-4k}\begin{bmatrix} n \\ 2k-1\end{bmatrix}.$$

Multiplying this by q^{2k^2+2k} and summing leads to

$$T_{n+2}-(1+q)T_{n+1}+(q-q^{2n+2})T_n=0.$$

The alternative form for the second family of Santos polynomials, as to be shown, is

$$\overline{T}_n=\sum_j q^{4j^2-3j}\begin{bmatrix} n \\ \lfloor\frac{n+2}{2}\rfloor-2j\end{bmatrix}_{q^2}.$$

The q-Vandermonde formula leads to

$$\begin{bmatrix} n \\ \lfloor \frac{n+2}{2} \rfloor - 2j \end{bmatrix}_{q^2} = \sum_k \begin{bmatrix} \lfloor \frac{n}{2} \rfloor \\ k+j \end{bmatrix}_{q^2} \begin{bmatrix} \lceil \frac{n}{2} \rceil \\ k+1-j \end{bmatrix}_{q^2} q^{2k^2+2k-2j^2+2j}$$

and therefore

$$\overline{T}_n = \sum_k q^{2k^2+2k} h(n,k),$$

with

$$h(n,k) := \sum_j q^{2j^2-j} \begin{bmatrix} \lfloor \frac{n}{2} \rfloor \\ k+j \end{bmatrix}_{q^2} \begin{bmatrix} \lceil \frac{n}{2} \rceil \\ k+1-j \end{bmatrix}_{q^2}.$$

Another elementary computation leads to

$$h(n+2,k) - (1+q)h(n+1,k) + q\,h(n,k) - q^{2n-2k} h(n,k-1).$$

Multiplying this by q^{2k^2+2k} and summing leads to

$$\overline{T}_{n+2} - (1+q)\overline{T}_{n+1} + (q - q^{2n+2})\overline{T}_n = 0,$$

as desired.

Additionally, we find $h(n,k) = \begin{bmatrix} n \\ 2k+1 \end{bmatrix}$.

References

[1] G. E. Andrews and J. P. Santos. Rogers-Ramanujan type identities for partitions with attached odd parts. *The Ramanujan Journal*, 1:91–99, 1997.

[2] George E. Andrews and Kimmo Eriksson. *Integer partitions*. Cambridge University Press, Cambridge, 2004.

[3] D. M. Bressoud. Some identities for terminating q-series. *Math. Proc. Cambridge Philos. Soc.*, 89(2):211–223, 1981.

[4] Robin Chapman. A new proof of some identities of Bressoud. *Int. J. Math. Math. Sci.*, 32(10):627–633, 2002.

[5] J. Cigler. Simple proofs of Bressoud's and Schur's polynomial versions of the Rogers-Ramanujan identities. *https://arxiv.org/abs/math/0701802*, 2007.

[6] S. Ole Warnaar. The generalized Borwein conjecture. II. Refined q-trinomial coefficients. *Discrete Mathematics*, 272:215–258, 2003.

Helmut Prodinger, Department of Mathematical Sciences, Stellenbosch University, 7602 Stellenbosch, South Africa

Email address: `hproding@sun.ac.za`