Bijections for a class of labeled plane trees

Nancy S. S. Gu [1,2,3]

Center for Combinatorics
Nankai University
Tianjin 300071
PR China

Helmut Prodinger [2]

Department of Mathematical Sciences
Stellenbosch University
7602 Stellenbosch
South Africa

Stephan Wagner [2]

Department of Mathematical Sciences
Stellenbosch University
7602 Stellenbosch
South Africa

Abstract

We consider plane trees whose vertices are given labels from the set $\{1, 2, \ldots, k\}$ in such a way that the sum of the labels along any edge is at most $k+1$; it turns out that the enumeration of these trees leads to a generalization of the Catalan numbers. We also provide bijections between this class of trees and $(k+1)$-ary trees as well as generalized Dyck paths whose step sizes are k (up) and 1 (down) respectively, thereby extending some classic results.

Key words: Labeled plane trees, k-ary trees, Dyck paths, bijections

1 Introduction

It is a classic result that *plane trees* with $n+1$ vertices and *binary trees* with n (internal) vertices are enumerated by the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. Plane trees are also known as ordered trees in the literature; the aforementioned binary trees, on the other hand, are sometimes called full or complete binary trees, since every internal vertex has exactly two children. If one considers the internal vertices only, one obtains so-called pruned binary trees, see for instance [9], whose internal vertices can either have two children, or only a left child, or only a right child. The simple bijection between these two classes of trees is known as the *natural correspondence* [14] or *rotation correspondence* [9] (Figure 1). It goes back to Harary, Prins and Tutte [13], its description was further simplified by de Bruijn and Morselt [6].

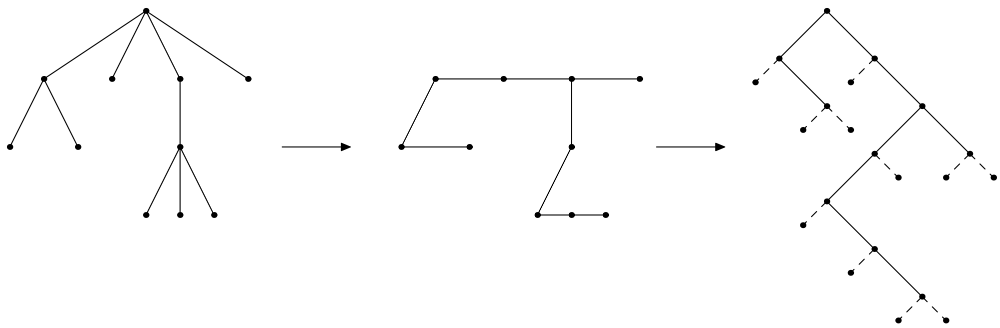

Fig. 1. The rotation correspondence.

In [12], a bijection was constructed between plane trees with $n+1$ vertices, labeled with two colors (black and white), such that the root is black, and no two vertices that are connected by an edge may be black, and ternary trees with n vertices.

It is thus a natural step to allow k colors, and construct a bijection between a suitable subclass of plane trees labeled by k colors, and $(k+1)$-ary trees. We address this question in the present paper. It turns out that the "right"

Email addresses: `gu@nankai.edu.cn` (Nancy S. S. Gu), `hproding@sun.ac.za` (Helmut Prodinger), `swagner@sun.ac.za` (Stephan Wagner).

[1] This paper was written while N. S. S. Gu was a visitor at the Center of Experimental Mathematics at the University of Stellenbosch. She thanks the center for its hospitality.

[2] This material is based upon work supported by the South African National Research Foundation under grant number 67215 (International science and technology agreement, SA/China).

[3] The first author was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, the National Science Foundation of China, and the Specialized Research Fund for the Doctoral Program of Higher Education of China (200800551042).

condition is to demand that the sum of the labels of any vertex and its parent may never exceed $k+1$, and that the root has to have color k. The aforementioned special case corresponds to white $= 1$, and black $= 2$. We will call such a tree a *k-plane tree*:

Definition 1. A k-plane tree is a plane tree whose vertices are given labels from the set $\{1, \dots, k\}$ in such a way that the sum of the labels along any edge is at most $k+1$.

In the following section, we use generating functions to enumerate k-plane trees; we even allow the root to have an arbitrary color i (instead of just k). Then a bijection between k-plane trees and a class of lattice paths is presented. The enumeration of these and many other families of lattice paths was treated in [1]. For $k = 1$, our bijection reduces to the classic *glove bijection* [2,4,18] between plane trees and Dyck paths.

Finally, we construct two different bijections between k-plane trees and $(k+1)$-ary trees, one of which is based on the correspondence between k-plane trees and lattice paths. For $k = 1$, both of them reduce to the aforementioned rotation correspondence, but they differ for $k \geq 2$.

2 Generating functions

Let $T_i(z)$ be the generating function for k-plane trees whose root is labeled i $(1 \leq i \leq k)$; in view of the definition of k-plane trees, we obtain a system of functional equations:

$$T_i(z) = \frac{z}{1 - \sum_{j=1}^{k+1-i} T_j(z)} \qquad \text{for all } i.$$

The easiest way to solve this system of equations is to use the substitution $z = \frac{v}{(1+v)^{k+1}}$ that is inspired by the Lagrange inversion formula [11,18] (compare also [5,8]): it turns out that $T_i(z) = \frac{v}{(1+v)^i}$. Indeed,

$$\frac{z}{1 - \sum_{j=1}^{k+1-i} \frac{v}{(1+v)^j}} = \frac{z}{1 - (1 - (1+v)^{-k-1+i})} = z(1+v)^{k+1-i} = \frac{v}{(1+v)^i}.$$

Since the power series for $T_1, T_2, \dots, T_k$ are uniquely determined by the functional equations, this shows that $T_i(z)$ has to be $\frac{v}{(1+v)^i}$. Now we can extract the n-th coefficient of T_i by means of contour integration:

$$\begin{aligned}
[z^n]T_i(z) &= \frac{1}{2\pi i}\oint \frac{v}{z^{n+1}(1+v)^i}\,dz \\
&= \frac{1}{2\pi i}\oint \frac{(1-kv)(1+v)^{(k+1)(n+1)}}{v^{n+1}(1+v)^{k+2}} \cdot \frac{v}{(1+v)^i}\,dv \\
&= \frac{1}{2\pi i}\oint \frac{(1-kv)(1+v)^{(k+1)n-i-1}}{v^n}\,dv \\
&= [v^{n-1}](1-kv)(1+v)^{(k+1)n-i-1} \\
&= \binom{(k+1)n-i-1}{n-1} - k\binom{(k+1)n-i-1}{n-2},
\end{aligned}$$

where the integrals are taken over suitably chosen contours around 0. Let us state this as a formal theorem:

Theorem 1. The number of k-plane trees with n vertices whose root is given the label i is precisely

$$\binom{(k+1)n-i-1}{n-1} - k\binom{(k+1)n-i-1}{n-2} = \frac{k-i+1}{kn-i+1} \cdot \binom{(k+1)n-i-1}{n-1}.$$

In particular, the special case $i = k$ yields

$$[z^n]T_k(z) = \frac{1}{k(n-1)+1} \cdot \binom{(k+1)(n-1)}{n-1},$$

i.e., one obtains a generalization of the Catalan numbers. It is well known that

$$\frac{1}{k(n-1)+1} \cdot \binom{(k+1)(n-1)}{n-1}$$

is also the number of $(k+1)$-ary trees with $n-1$ internal vertices or the number of lattice paths comprising of $n-1$ upsteps of size k and $k(n-1)$ downsteps of size 1 that start at 0 and stay above the x-axis. In the following two sections, we construct bijections between these objects and k-plane trees whose root is labeled k. These bijections generalize the classic bijections between plane trees and binary trees and between plane trees and Dyck paths.

Finally, it should also be mentioned that one obtains yet another generalization of the Catalan numbers if one takes the sum over all i: since

$$T(z) = \sum_{i=1}^{k} T_i(z) = 1 - \frac{z}{T_1(z)} = 1 - (1+v)^{-k},$$

one obtains

$$
\begin{aligned}
[z^n]T(z) &= \frac{1}{2\pi i}\oint \frac{1-(1+v)^{-k}}{z^{n+1}}\,dz \\
&= \frac{1}{2\pi i}\oint \frac{(1-kv)(1+v)^{(k+1)(n+1)}}{v^{n+1}(1+v)^{k+2}}\cdot\Big(1-(1+v)^{-k}\Big)\,dv \\
&= \frac{1}{2\pi i}\oint \frac{(1-kv)((1+v)^k-1)(1+v)^{(k+1)(n-1)}}{v^{n+1}}\,dv \\
&= [v^n](1+v)^{(k+1)n-1} - [v^n](1+v)^{(k+1)(n-1)} \\
&\quad - k[v^{n-1}](1+v)^{(k+1)n-1} + k[v^{n-1}](1+v)^{(k+1)(n-1)} \\
&= \binom{(k+1)n-1}{n} - \binom{(k+1)(n-1)}{n} \\
&\quad - k\binom{(k+1)n-1}{n-1} + k\binom{(k+1)(n-1)}{n-1} \\
&= \frac{k}{n}\binom{(k+1)(n-1)}{n-1}.
\end{aligned}
$$

Hence we have the following theorem:

Theorem 2. The total number of all k-plane trees with n vertices is

$$\frac{k}{n}\binom{(k+1)(n-1)}{n-1} = \frac{1}{n-1}\binom{(k+1)(n-1)}{n}.$$

Apparently, this generalization of the Catalan numbers does not appear very often in the literature. Sloane's Encyclopedia of Integer Sequences [17] provides a few references in the case $k = 2$ (such as [10,16]), and the general case appears in [3] (even in a slightly more general form), but it seems that there are not many known enumeration problems that lead to these numbers for general k.

3 Bijection between k-plane trees and generalized Dyck paths

Consider lattice paths that do not go below the x-axis and consist of n upsteps of size k and kn downsteps of size 1. It is easy to show, for instance by means of the *cycle lemma* of Dvoretzky and Motzkin [7], that the number of such lattice paths that stay above the x-axis is exactly the generalized Catalan number $\frac{1}{kn+1}\binom{(k+1)n}{n}$. Let us describe how such a lattice path can be constructed from a k-plane tree whose root is labeled k.

One proceeds as in the classic glove bijection [2,4,18]: starting to the left of the root of a given tree T, we move around the tree, always moving away from the root on the left hand side of an edge and towards the root on the right hand side of an edge. Each of the edges that we encounter corresponds to one upstep and k downsteps as follows: whenever we move along an edge away from the root, and the terminal vertex of this edge has label j, then we add $j-1$ downsteps, followed by an upstep, to the lattice path. On the way back, we add the remaining $k-j+1$ downsteps to the lattice path when we move along this edge. Figure 2 shows a few steps of this procedure in the case $k=3$; the complete lattice path that corresponds to the given tree is shown in Figure 3.

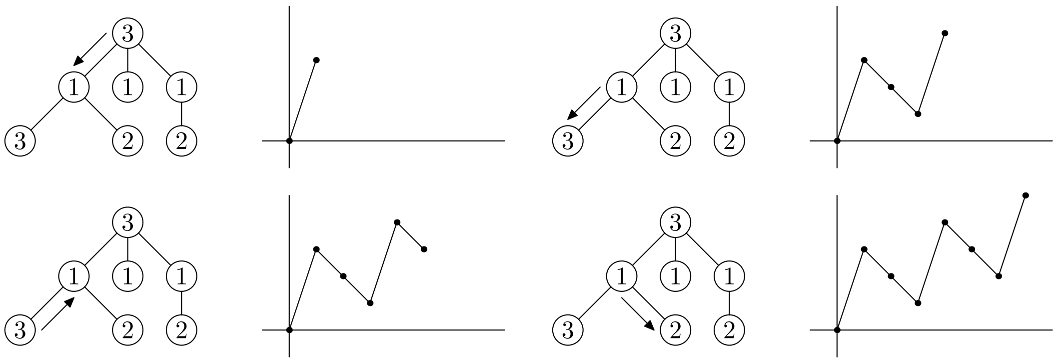

Fig. 2. Bijection between k-plane trees and lattice paths.

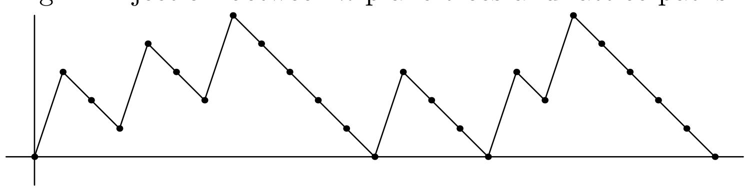

Fig. 3. The complete lattice path.

Let us prove that this is indeed a bijection. In the following, $\ell(v)$ denotes the label of a vertex v. First of all, note that we can assign a level in the lattice path to every vertex of T: the level of the root is 0, and the level of a non-root vertex v is $k+1-\ell(v)$ plus the level of its parent. Therefore, the levels are strictly increasing as one moves away from the root, and since all children of the root bear the label 1, their level is k. So we can conclude that every non-root vertex has level $\geq k$. When moving along an edge, one can never add more than $k-1$ downsteps in the lattice path, and so one will never fall below the x-axis when one is moving along a edge away from the root; note that this is also true for edges that start at the root, since all such edges correspond to 0 downsteps, followed by an upstep. When one is moving towards the root, the corresponding part of the lattice path merely consists of downsteps that end at a nonnegative level, and so the lattice path will also always stay above the x-axis in this case.

The condition on the labels of a k-plane tree guarantees that the reconstruction is unique: suppose that we have reconstructed the tree from the lattice path up to a certain point that corresponds to a vertex v whose label is $\ell(v) = j$. Any potential child of v has label $\leq k+1-j$ and will thus correspond to at most $k-j$ downsteps, followed by an upstep. Hence, if the following segment of the lattice path begins with a sequence of $i \leq k-j$ downsteps, followed by an upstep, we add a child w to v and assign the label $i+1$ to it. Now we continue the process from w, etc. If, on the other hand, we encounter $k+1-j$ or more downsteps, then we move towards the root from v, which corresponds to exactly $k+1-j$ downsteps. Then we continue from v's parent.

If we wanted to reconstruct the 3-plane tree from the lattice path shown in Figure 3, we would proceed as follows: the first upstep corresponds to the leftmost child of the root, whose label must be 1. This is followed by two downsteps, followed by an upstep. Hence we attach a vertex that is labeled 3. Now we encounter two downsteps again, but since it is impossible to add another vertex labeled 3 by our restrictions, we have to move back towards the root again. Then we are left with one downstep, followed by an upstep, which corresponds to a vertex whose label is 2, etc.

Let us mention that the same bijection can also be applied to k-plane trees whose root is not labeled k; in this case, the corresponding lattice paths have the property that they always stay above the line $y = j - k$, where j is the root's label. Alternatively, one can think of lattice paths that start at $(0, k-j)$ (and also end on the line $y = k - j$) and stay above the x-axis. Altogether, the bijection shows that the number of lattice paths consisting of n upsteps of size k and kn downsteps of size 1 which start at $(0, i)$ for some $0 \leq i < k$ and stay above the x-axis is exactly the generalized Catalan number that we encountered in Theorem 2, namely $\frac{k}{n+1}\binom{(k+1)n}{n}$. Note also that one simply obtains the classic glove bijection in the case that $k = 1$.

4 Bijections between k-plane trees and $(k+1)$-ary trees

In this section we present two bijections between k-plane trees whose root is labeled k and $(k+1)$-ary trees. The first one is essentially based on the bijection presented in the previous section, the second one provides an interesting alternative approach, even though it is more complicated to formulate.

There is a simple bijection between the generalized Dyck paths discussed in the previous section and $(k+1)$-ary trees that is (in essence) due to Kuich [15]: split a path with n upsteps of size k and kn downsteps into segments that consist of an upstep and all downsteps immediately following it. The lengths of these segments form a sequence $a_1, a_2, \ldots, a_n$. Now construct a $(k+1)$-ary tree as follows: starting with $k+1$ edges attached to the root, visit leaves in preorder (depth first, from left to right, thereby moving around the tree as in the glove bijection) and attach $k+1$ new leaves to the a_ith leaf visited at step i, $1 \le i \le n-1$ (the last term of the sequence, which is uniquely determined by the others, is ignored). The reverse procedure is immediate. Figure 4 shows the construction of the tree corresponding to the path in Figure 3, whose associated sequence is (3,3,6,4,2,6).

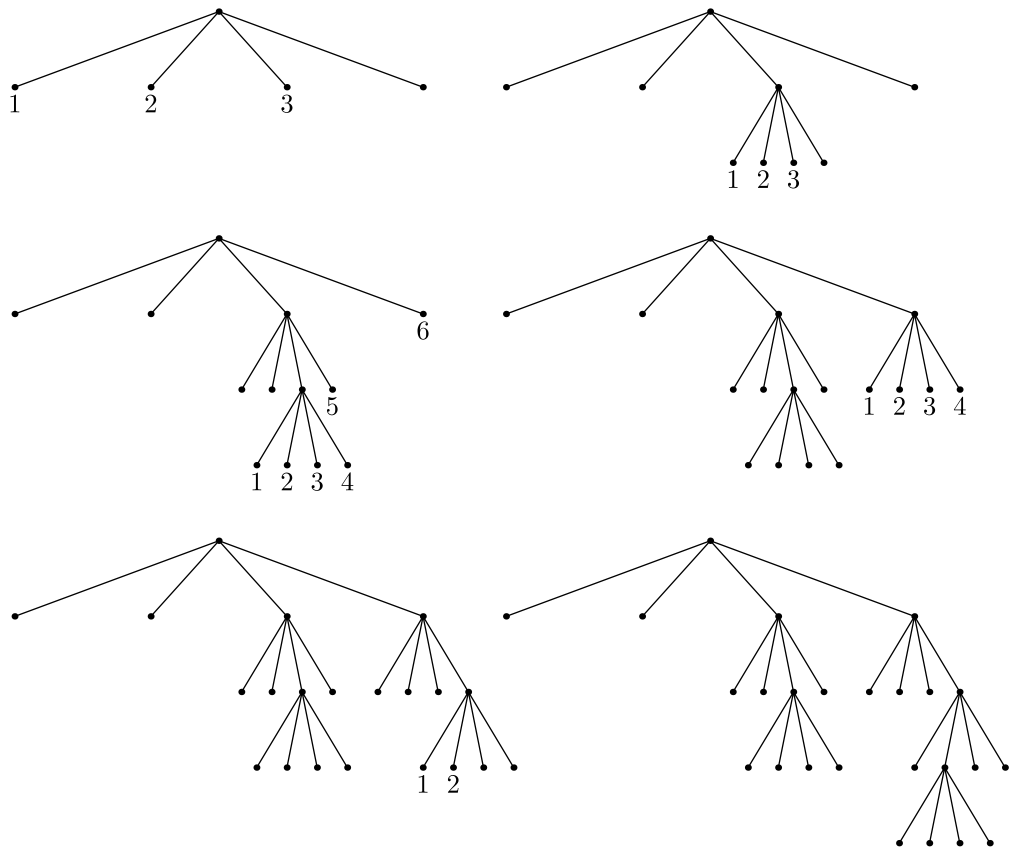

Fig. 4. Constructing a $(k+1)$-ary tree from a lattice path.

Another approach that leads to the same bijection is based on *Lukasiewicz codes*, see [9]: moving around a $(k+1)$-ary tree in counterclockwise direction as before, record the outdegree of every vertex when it is visited for the first time. This yields a sequence whose terms are $k+1$ or 0. Subtracting 1 from every element of the sequence, one obtains the sequence of step sizes of the corresponding path. Again the inverse bijection is also simple.

Yet another essentially equivalent approach is to replace each big upstep of size k by $k+1$ small upsteps of size 1, followed by a downstep, to obtain a Dyck path whose maximal runs of upsteps consist of exactly $k+1$ steps. Then one can apply (a generalization of) the procedure described by de Bruijn and Morselt [6].

The composition of the bijection between generalized Dyck paths and $(k+1)$-ary trees and the bijection described in the previous section clearly yields a bijection between $(k+1)$-ary trees and k-plane trees. However, it can also be described directly, which is done in the following. We exhibit how a $(k+1)$-ary tree is constructed from a k-plane tree, the reverse step is immediate. As before, we move around the given k-plane tree in counterclockwise direction. This is done simultaneously for the resulting $(k+1)$-ary tree, which emerges on the way (at the beginning, it only consists of the root). Whenever we move away from the root along an edge that leads to a vertex labeled ℓ, we move ℓ leaves forward (in preorder) in the $(k+1)$-ary tree and attach $(k+1)$ new leaves to the leaf that we reach (at the beginning, this means that we add new leaves to the root). On the other hand, when we move from a vertex labeled ℓ towards the root in the k-plane tree, then we move $k+1-\ell$ leaves forward, but without attaching new leaves at the end. Figure 5 shows the first few steps for the example of Figure 2; note that the corresponding $(k+1)$-ary tree evolves in essentially the same way as in Figure 4.

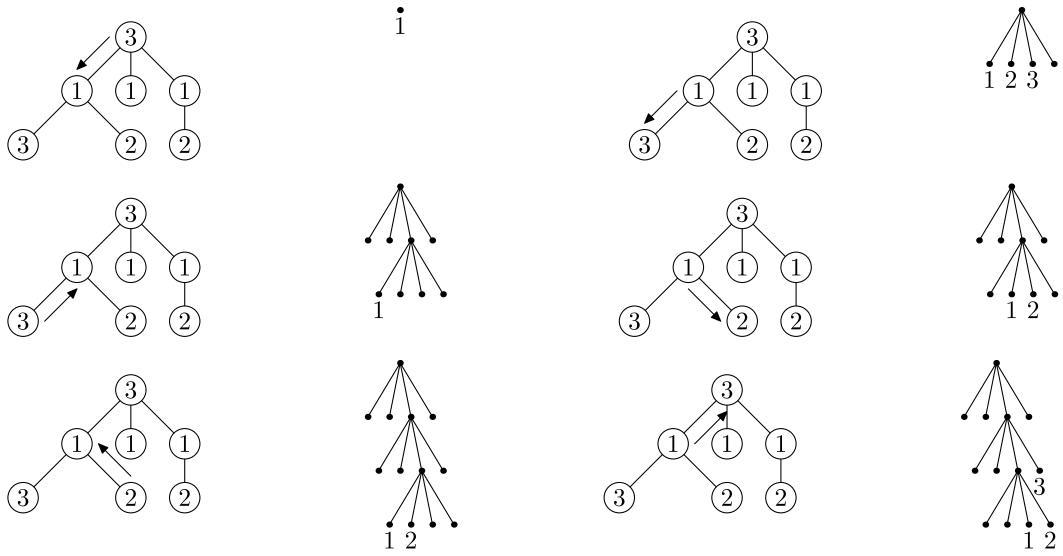

Fig. 5. The first bijection between k-plane trees and $(k+1)$-ary trees.

The condition that the root bears label k ensures that all its children are labeled 1, which is necessary since there is only one leaf where new vertices can be attached at the beginning (and every time one returns to the root). In the general case that the root's label is i, one can adjust the bijection by starting with a collection of $k+1-i$ roots. Then one ends up with a sequence of $k+1-i$ $(k+1)$-ary trees (possibly only consisting of the root). Alternatively, one can regard this collection of $k+1-i$ $(k+1)$-ary trees as a single tree with the property that the root has outdegree $k+1-i$, while all other internal vertices have outdegree $k+1$. Note also that this agrees with the generating functions found in Section 2: the generating function for such trees is exactly

$$z \cdot \left(\frac{1}{z} \cdot \frac{v}{(1+v)^k}\right)^{k+1-i} = \frac{v}{(1+v)^{k+1}} \cdot (1+v)^{k+1-i} = \frac{v}{(1+v)^i}.$$

Figure 6 shows an example of this construction in the case $k = 3$.

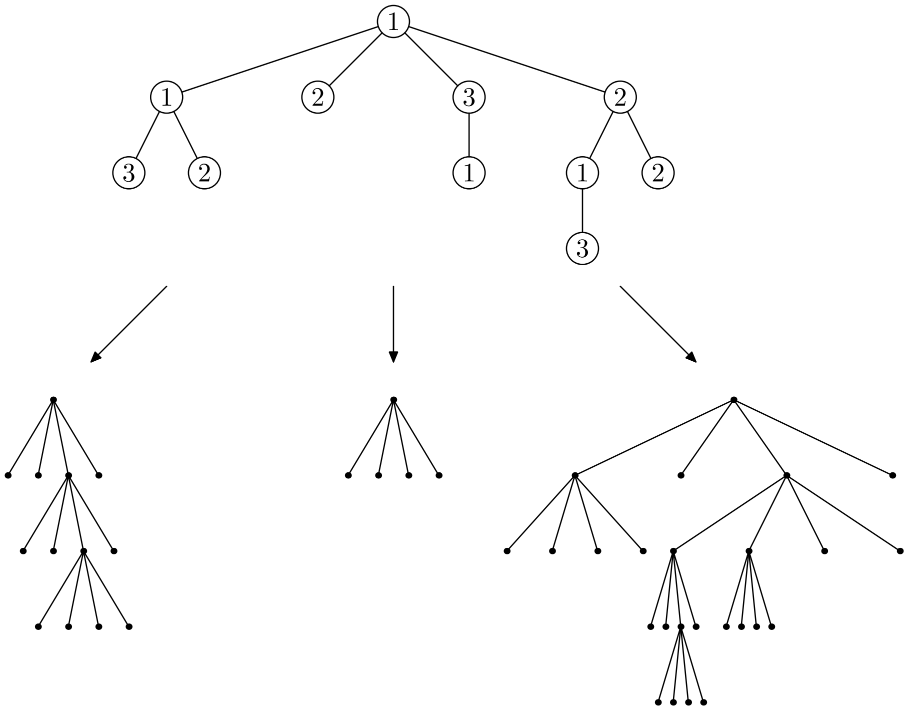

Fig. 6. The case of general root labels.

Let us now present the second bijection; for $k = 1$, it is identical to the rotation correspondence, but it differs from the first bijection for $k \geq 2$ as well as from the bijection presented in [12]. First it is explained how a $(k+1)$-ary tree is constructed from a k-plane tree. The reverse process will follow almost automatically.

From k-plane trees to $(k+1)$-ary trees

We call a vertex v of a k-plane tree a left descendant of u if there is a sequence of vertices $u = u_1, u_2, \ldots, u_r = v$ such that u_{j+1} is the leftmost child of u_j for every j.

Let a k-plane tree T with $n+1$ vertices be given; we construct a $(k+1)$-ary tree T^* with n internal vertices by associating a vertex v^* with every non-root vertex v of T. We will use the following definition: if the label of v's parent in T is j, then we call the j leftmost positions where a child can be attached to v^* the α-positions, and the remaining $k+1-j$ positions of attachment the β-positions (Figure 7 shows an example in the case $k=5$). It will become clear from the construction that follows that the α-positions are reserved for vertices associated to left descendants of v, while the β-positions are reserved for v's right sibling (if there is one) and the left descendants of this sibling.

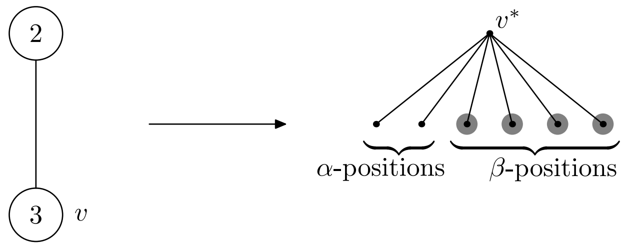

Fig. 7. α-and β-positions (the latter are indicated by gray shades).

Let us now describe how the tree T^* is constructed: the leftmost child of the root of T is associated with the root of T^*. The remaining vertices of T are traversed in a depth-first way, according to the following rules:

- If a vertex v is not a leftmost child and u is its left sibling, then v^* is attached to u^* at the $\ell(v)$-th position from the right (note that this is a β-position, since the label $\ell(v)$ can at most be $k+1-j$ if the label of the common parent of u and v is j). The $\ell(v)-1$ positions to the right of v^* will remain unoccupied for the rest of the process. See Figure 8 for an example in the case $k=3$.

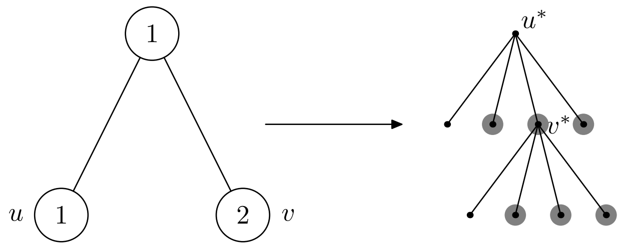

Fig. 8. Handling right siblings. Gray shades indicate β-positions.

- If a vertex v is a leftmost child, then consider the first ancestor of v that is not a leftmost child; in other words, let $u_1, u_2, \ldots, u_r = v$ be a sequence of vertices such that u_{j+1} is the leftmost child of u_j for every j and u_1 is not a leftmost child (and thus either the root of T or a right sibling of some other vertex). Now we have to distinguish two subcases (however, they are quite similar):
  - Assume that u_1 is the root of T; in this case, let P be the set consisting of the α-positions of all u_j^*, $2 \le j \le r-1$. Of all the positions in P, we consider those that follow the last position that is already occupied (if any; otherwise, we just consider all of them), counting from top to bottom and from right to left. The vertex $v^* = u_r^*$ is attached to the $\ell(u_r)$-th of these positions, again counting from top to bottom and from right to left.

Of course we have to make sure that this is actually always possible: if $r = 3$, then there are $\ell(u_1) = k$ positions, and there are indeed exactly k possible labels for $\ell(u_3)$ (since we must necessarily have $\ell(u_2) = 1$). Note also that $k = k + 1 - \ell(u_2)$.

Now we proceed inductively to show that there are always precisely $k + 1 - \ell(u_{r-1})$ positions of attachment for u_r^*, which is also the number of possible labels for u_r: when u_r^* is attached, we lose $\ell(u_r)$ possible positions of attachment (the position where u_r^* is attached, but also $\ell(u_r) - 1$ previously empty positions to the right and above it); on the other hand, u_r^* has $\ell(u_{r-1})$ α-positions by definition, which are added. This gives us exactly

$$k + 1 - \ell(u_{r-1}) - \ell(u_r) + \ell(u_{r-1}) = k + 1 - \ell(u_r)$$

possible positions for u_{r+1}, as desired. Figure 9 shows several steps of this procedure in the case $k = 3$; potential positions of attachment are marked by a circle; β-positions are indicated by a gray mark. Dashed lines indicate possible connections to other vertices.

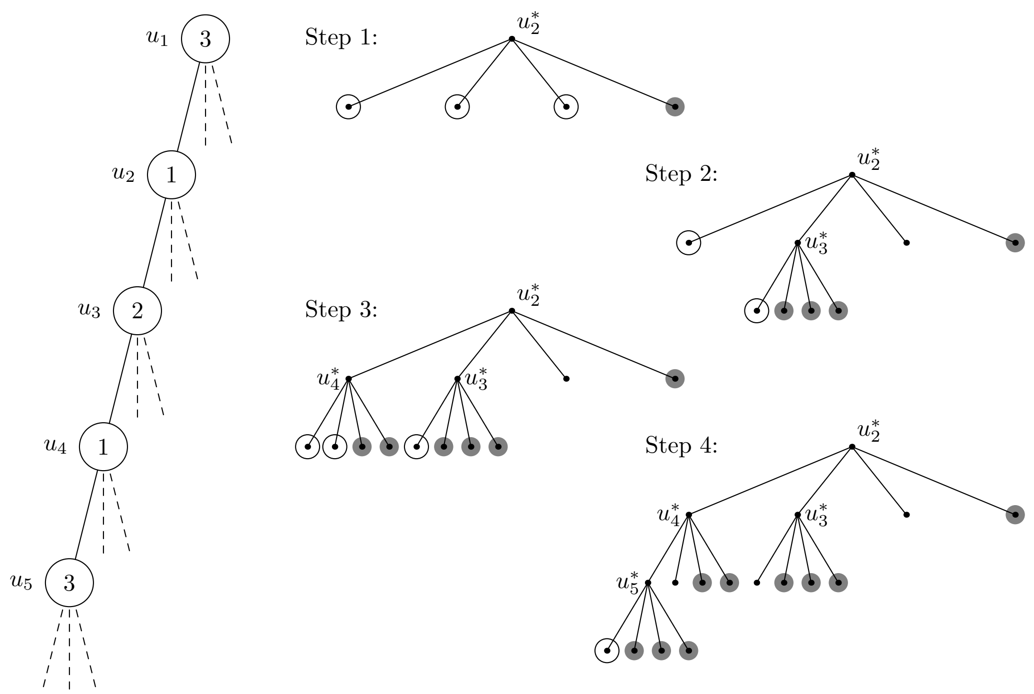

Fig. 9. Handling left descendants of the root. Gray shades indicate β-positions; circles mark potential positions of attachment for the following vertex.

· If u_1 is the right sibling of some vertex w and u_0 is the common parent of u_1 and w, we proceed in a similar way: in this case, let P be the set consisting of the β-positions of w^* together with the α-positions of all u_j^*, $1 \leq j \leq r - 1$. Of all the positions in P, we consider those that follow the last position that is already occupied, counting from top to bottom and from right to left. Now the vertex u_r^* is attached to the $\ell(u_r)$-th of these positions, as in the previous case.

Again it is easy to show that there is exactly the right number of such positions available: if $r = 1$, then w^* can provide $k + 1 - \ell(u_0)$ empty β-positions; this is also exactly the number of possible labels for u_1. When u_r^* is attached, we lose $\ell(u_r)$ possible positions and gain $\ell(u_{r-1})$, resulting in $k+1-\ell(u_r)$ positions, as in the first case. Figure 10 shows several steps of this procedure in the case $k = 3$.

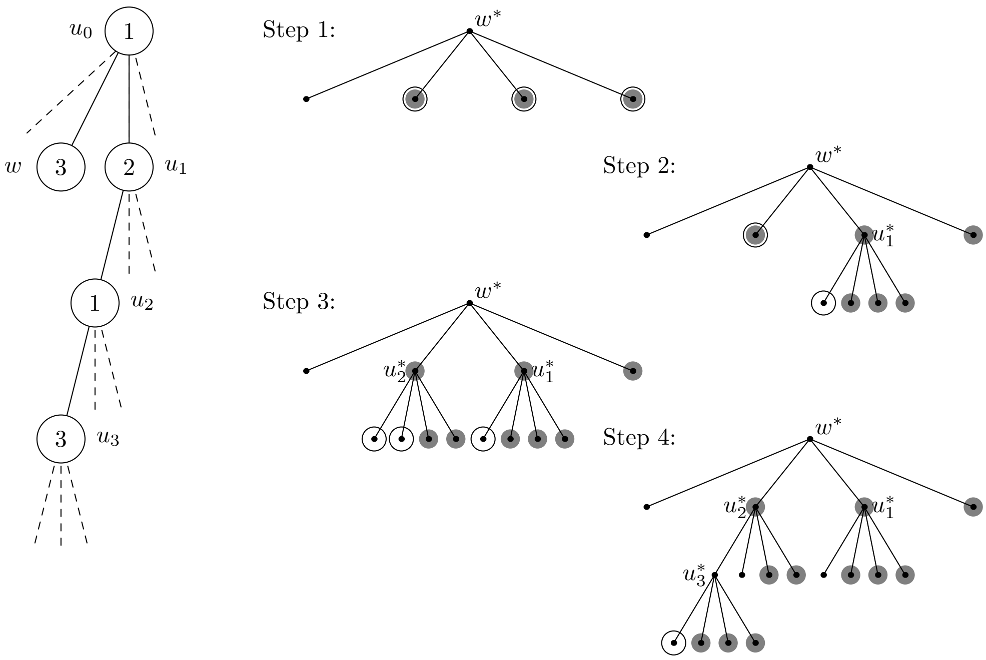

Fig. 10. Handling right siblings and their left descendants. Gray shades indicate β-positions; circles mark potential positions of attachment for the following vertex.

Let us remark that the vertices do not necessarily have to be traversed depth-first (breadth-first would be possible too, for instance), as long as all vertices are processed after their parents and left siblings. However, the depth-first algorithm seemed to be most canonical to us.

From $(k + 1)$-ary trees to k-plane trees

It is not difficult to reverse the process: once a vertex v in T has been reconstructed, it is also possible to determine the possible positions of attachment for its leftmost child (if v has any children). The first of these positions that is occupied (counting from top to bottom and from right to left, as in the construction described above) corresponds to the leftmost child (which allows us to reconstruct the label of this child); if none of the positions is occupied, then v has no children. The same applies to the potential right sibling of v: the rightmost occupied β-position of v^* corresponds to the right sibling of v.

Consider, for instance, the situation in Figure 9: suppose that u_2, u_3 and u_4 have already been reconstructed. This allows us to determine the possible positions of attachment for a left child of u_4. We see that the first of these positions that is occupied (counting top-down and right-left) is the third position; this shows that u_4 has at least one child and that the leftmost child has label 3.

Likewise, consider the situation in Figure 10: suppose that u_0 and w have already been reconstructed. We can thus determine the possible positions of attachment for a right sibling of w: since the second of these positions is the first one that is occupied, we know that there is a right sibling and that it has label 2.

Figure 11 shows a complete 3-plane tree and the corresponding 4-ary tree. Let us finally remark that one obtains the classic bijection between plane trees and binary trees in the case that $k = 1$: in this case, all labels are 1, so there is always precisely one possible position for each vertex. Vertices corresponding to leftmost children are attached on the left hand side, vertices corresponding to right siblings are attached on the right hand side.

Let us briefly describe how this construction can be extended to the case that the root's label is an arbitrary number between 1 and k. There are two reasons why the root has to have label k in our construction:

- It makes the label of the root's leftmost child (vertex u_2 in Figure 11) unique, which could otherwise not be reconstructed from the $(k+1)$-ary tree.
- As a consequence of the fact that the root's leftmost child has label 1, it is ensured that the number of α-positions of the vertex associated to it (vertex u_2^* in Figure 11) is exactly the number of possible labels for its own leftmost child (if there is one; in Figure 11, this is vertex u_3).

Both conditions remain satisfied in the case that the root is labeled i if we impose the additional restriction that the root's leftmost child (let us denote it by v) must get label $k + 1 - i$. Then it is clear that the first condition (reconstructability of v's label) holds, and we only have to check the second condition: but this is also easy, since v^* has exactly i α-positions under our assumptions, which is also the number of possible labels for v's leftmost child (note that $i = k + 1 - (k + 1 - i)$). Hence our procedure can still be applied, and it is also still uniquely reconstructable.

If the root's leftmost child is not necessarily labeled $k+1-i$, one can proceed as follows: find the root's first child that is labeled $k+1-i$ (from left to right, if there is such a child), and denote it by v. Now we just consider that part of the k-plane tree that is formed by the branch that corresponds to v and all branches to the right of it. As described before, one can uniquely associate a

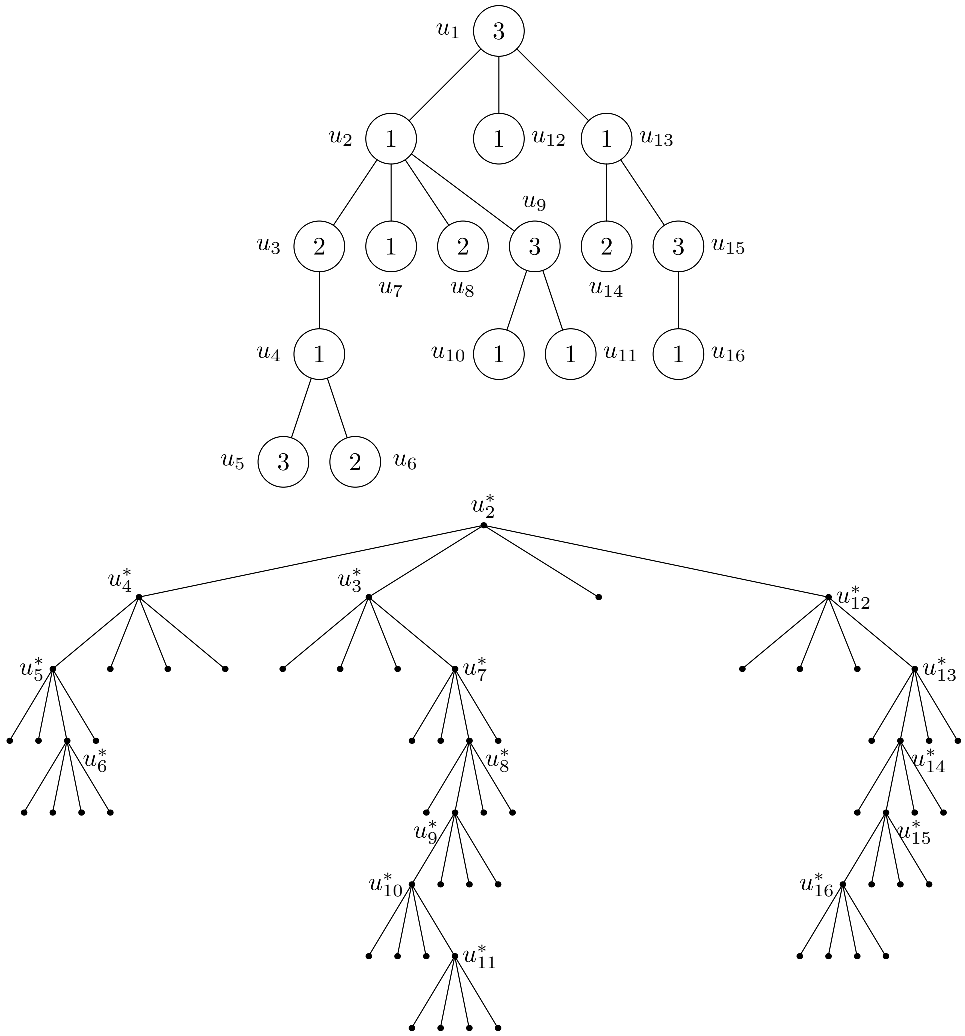

Fig. 11. A complete example.

$(k+1)$-ary tree to it. Now remove all these branches and continue with the remaining tree.

All the root's children must now have labels $\leq k-i$, and so we may replace the root's label by $i+1$ and repeat the process (find the root's leftmost child labeled $k-i$, etc.). This can be done $k+1-i$ times, and the result is again a sequence of $k+1-i$ $(k+1)$-ary trees, which is what we also obtained from the first bijection. Figure 12 shows an example in the case $k=3$.

5 Conclusion

Our bijections extend the well-known bijections between plane trees and binary trees resp. plane trees and Dyck paths. While the generalized Catalan numbers $\frac{1}{k(n-1)+1}\binom{(k+1)(n-1)}{n-1}$ (which enumerate k-plane trees whose root is la-

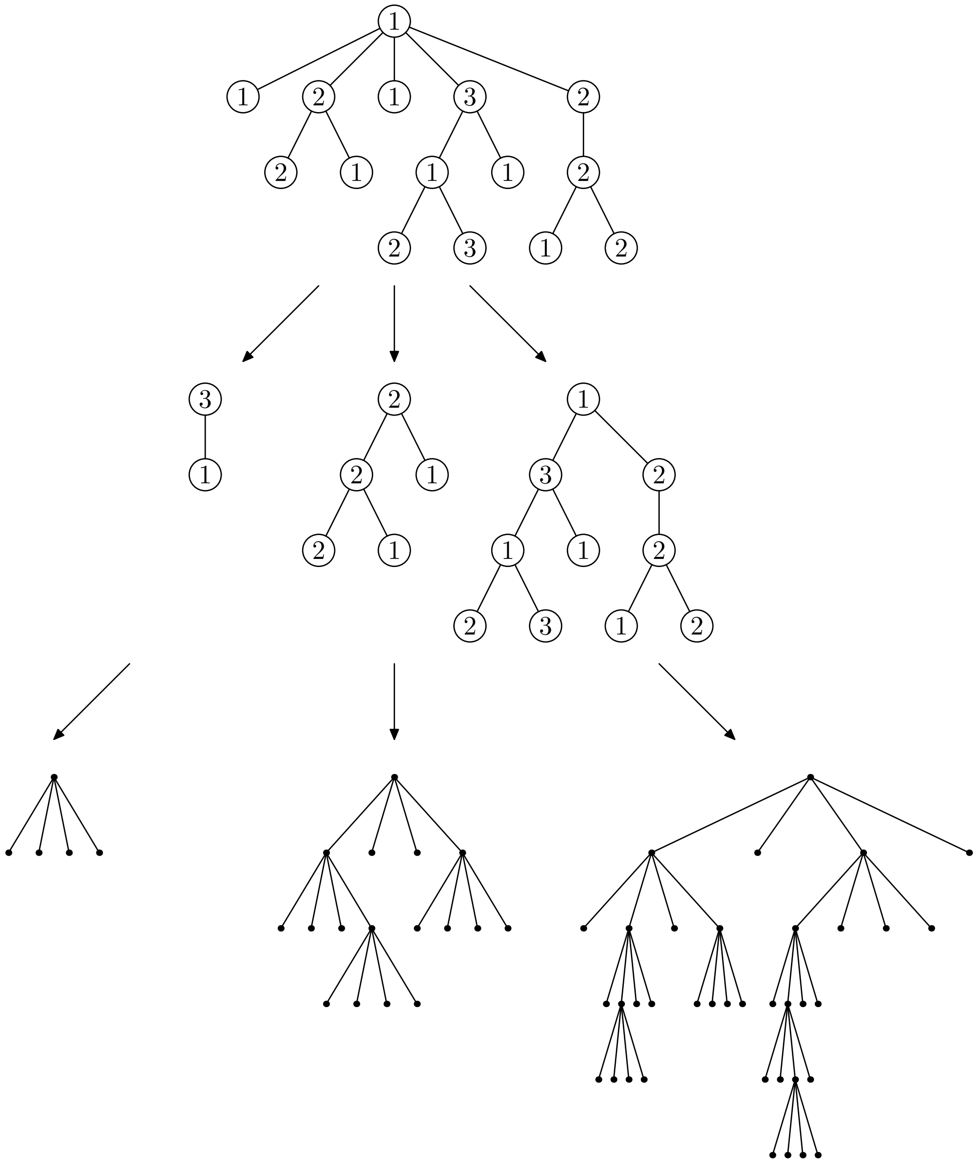

Fig. 12. The bijection in the case that the root is not labeled k.

beled k) occur quite frequently in the literature, this does not seem to be the case for the numbers $\frac{k}{n}\binom{(k+1)(n-1)}{n-1}$, which enumerate all k-plane trees. It would be interesting to see other enumeration problems that lead to these numbers.

The class of k-plane trees, as defined in this paper, also provides some possibilities for further investigations: for instance, one could ask for bijections between r-tuples of k-plane trees whose roots are labeled $i_1, i_2, \dots, i_r$ and r-tuples of k-plane trees whose roots are labeled $j_1, j_2, \dots, j_r$, provided that

$$i_1 + i_2 + \cdots + i_r = j_1 + j_2 + \cdots + j_r = s,$$

since both have generating function $\frac{v^r}{(1+v)^s}$. The correspondence between k-plane trees with arbitrary root labels and tuples of $(k+1)$-ary trees (see the end of Section 4) clearly provides such a bijection, but it is not very direct.

Finally, one can certainly modify the definition of k-plane trees by imposing other restrictions on pairs of labels along an edge. It is conceivable that appropriate conditions will lead to interesting counting problems as well.

Acknowledgment

We are very grateful to two anonymous referees for their valuable suggestions.

References

[1] C. Banderier, P. Flajolet, Basic analytic combinatorics of directed lattice paths, Theoret. Comput. Sci. 281 (2002) 37–80.

[2] D. Callan, Some bijections and identities for the Catalan and Fine numbers, Sém. Lothar. Combin. 53 (2004/06) Art. B53e, 16 pp. (electronic).

[3] D. Callan, A combinatorial interpretation of $\frac{j}{n}\binom{kn}{n+j}$, arXiv math/CO 0604471 (2006).

[4] W. Y. C. Chen, N. Y. Li, L. W. Shapiro, The butterfly decomposition of plane trees, Discrete Appl. Math. 155 (17) (2007) 2187–2201.

[5] N. G. de Bruijn, D. E. Knuth, S. O. Rice, The average height of planted plane trees, in: Graph theory and computing, Academic Press, New York, 1972, pp. 15–22.

[6] N. G. de Bruijn, B. J. M. Morselt, A note on plane trees, J. Combinatorial Theory 2 (1967) 27–34.

[7] A. Dvoretzky, T. Motzkin, A problem of arrangements, Duke Math. J. 14 (1947) 305–313.

[8] P. Flajolet, M. Noy, Analytic combinatorics of non-crossing configurations, Discrete Math. 204 (1-3) (1999) 203–229.

[9] P. Flajolet, R. Sedgewick, Analytic combinatorics, Cambridge University Press, Cambridge, 2009.

[10] I. M. Gessel, G. Xin, The generating function of ternary trees and continued fractions, Electron. J. Combin. 13 (1) (2006) Research Paper 53, 48 pp. (electronic).

[11] I. P. Goulden, D. M. Jackson, Combinatorial enumeration, Dover Publications Inc., Mineola, NY, 2004.

[12] N. S. S. Gu, H. Prodinger, Bijections for 2-plane trees and ternary trees, European J. Combin. 30 (4) (2009) 969–985.

[13] F. Harary, G. Prins, W. T. Tutte, The number of plane trees, Nederl. Akad. Wetensch. Proc. Ser. A 67=Indag. Math. 26 (1964) 319–329.

[14] D. E. Knuth, The Art of Computer Programming, vol. 1: Fundamental Algorithms, Addison-Wesley, 1968, third edition, 1997.

[15] W. Kuich, Languages and the enumeration of planted plane trees, Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math. 32 (1970) 268–280.

[16] J. Quaintance, Combinatoric enumeration of two-dimensional proper arrays, Discrete Math. 307 (15) (2007) 1844–1864.

[17] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `http://www.research.att.com/~njas/sequences`.

[18] R. P. Stanley, Enumerative combinatorics. Vol. 2, vol. 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999.