A CLOSED FORMULA FOR THE GENERATING FUNCTION OF p-BERNOULLI NUMBERS: AN ELEMENTARY PROOF

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Abstract. For a two parameter family of Bernoulli numbers $B_{n,p}$ the exponential generating function is derived by elementary methods.

1. Introduction

The following recursion for a two paramenter family of Bernoulli numbers is given in [3],

$$B_{n+1,p} = pB_{n,p} - \frac{(p+1)^2}{p+2} B_{n,p+1} \quad \text{for} \quad n, p \geq 0. \tag{1}$$

In terms of exponential generating functions

$$f_p(t) := \sum_{n \geq 0} B_{n,p} \frac{t^n}{n!},$$

recursion (1) translates into

$$f_p'(t) = pf_p(t) - \frac{(p+1)^2}{p+2} f_{p+1}(t).$$

The closed formula that follows is the main result of [2].

Theorem 1. *For $p \geq 0$*

$$f_p(t) = \sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = \frac{(p+1)(t-H_p)e^{pt}}{(e^t-1)^{p+1}} + (p+1) \sum_{k=1}^{p} \binom{p}{k} \frac{H_k}{(e^t-1)^{k+1}}, \tag{2}$$

where H_n is the harmonic numbers defined in [1]*:*

$$H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

We provide a shorter proof of this theorem using elementary methods.

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2. Proof

For $p = 0$ we have

$$\sum_{n=0}^{\infty} B_{n,0}\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = \frac{t}{e^t-1}$$

and

$$\frac{(0+1)(t-H_0)e^0}{(e^t-1)^{0+1}} + (0+1)\sum_{k=1}^{0}\binom{0}{k}\frac{H_k}{(e^t-1)^{k+1}} = \frac{t}{e^t-1}.$$

Therefore Equation (2) holds for $p = 0$.

Now, assume that it holds for some p. Then

$$f_p(t) = \frac{(p+1)(t-H_p)e^{pt}}{(e^t-1)^{p+1}} + (p+1)\sum_{k=1}^{p}\binom{p}{k}\frac{H_k}{(e^t-1)^{k+1}}.$$

It follows that

$$\begin{aligned}
pf_p(t) - f_p'(t) &= p(p+1)\frac{(t-H_p)e^{pt}}{(e^t-1)^{p+1}} + p(p+1)\sum_{k=1}^{p}\binom{p}{k}\frac{H_k}{(e^t-1)^{k+1}} \\
&\quad - (p+1)\frac{e^{pt}}{(e^t-1)^{p+1}} - p(p+1)\frac{(t-H_p)e^{pt}}{(e^t-1)^{p+1}} + (p+1)^2\frac{(t-H_p)e^{(p+1)t}}{(e^t-1)^{p+2}} \\
&\quad + (p+1)\sum_{k=1}^{p}\binom{p}{k}(k+1)\frac{H_k e^t}{(e^t-1)^{k+2}} \\
&= p(p+1)\sum_{k=1}^{p}\binom{p}{k}\frac{H_k}{(e^t-1)^{k+1}} - (p+1)\frac{e^{pt}}{(e^t-1)^{p+1}} + (p+1)^2\frac{(t-H_p)e^{(p+1)t}}{(e^t-1)^{p+2}} \\
&\quad + (p+1)\sum_{k=1}^{p}\binom{p}{k}(k+1)\frac{H_k}{(e^t-1)^{k+1}} + (p+1)\sum_{k=1}^{p}\binom{p}{k}(k+1)\frac{H_k}{(e^t-1)^{k+2}} \\
&= -(p+1)\frac{e^{pt}}{(e^t-1)^{p+1}} + (p+1)^2\frac{(t-H_p)e^{(p+1)t}}{(e^t-1)^{p+2}} + p(p+1)\sum_{k=1}^{p}\binom{p}{k}\frac{H_k}{(e^t-1)^{k+1}} \\
&\quad + (p+1)\sum_{k=1}^{p}\binom{p}{k}(k+1)\frac{H_k}{(e^t-1)^{k+1}} + (p+1)\sum_{k=1}^{p+1}\binom{p}{k-1}k\frac{H_k-\frac{1}{k}}{(e^t-1)^{k+1}} \\
&= -(p+1)\frac{e^{pt}}{(e^t-1)^{p+1}} + (p+1)^2\frac{(t-H_p)e^{(p+1)t}}{(e^t-1)^{p+2}} \\
&\quad + (p+1)\sum_{k=1}^{p}\frac{H_k}{(e^t-1)^{k+1}}\left[p\binom{p}{k} + \binom{p}{k}(k+1) + \binom{p}{k-1}k\right]
\end{aligned}$$

$$
\begin{aligned}
&- (p+1)\sum_{k=1}^{p}\binom{p}{k-1}\frac{1}{(e^t-1)^{k+1}} + (p+1)^2\frac{H_p}{(e^t-1)^{p+2}}\\
&= -\frac{(p+1)e^{pt}}{(e^t-1)^{p+1}} + (p+1)^2\frac{(t-H_{p+1})e^{(p+1)t}}{(e^t-1)^{p+2}} + \frac{(p+1)e^{(p+1)t}}{(e^t-1)^{p+2}}\\
&\quad + (p+1)^2\sum_{k=1}^{p+1}\frac{H_k}{(e^t-1)^{k+1}}\binom{p+1}{k} - (p+1)^2\frac{H_{p+1}}{(e^t-1)^{p+2}}\\
&\quad - \frac{(p+1)e^{pt}}{(e^t-1)^{p+2}} + (p+1)^2\frac{H_{p+1}}{(e^t-1)^{p+2}}\\
&= -\frac{(p+1)e^{pt}}{(e^t-1)^{p+1}} + \frac{(p+1)e^{(p+1)t}}{(e^t-1)^{p+2}} - \frac{(p+1)e^{pt}}{(e^t-1)^{p+2}}\\
&\quad + (p+1)^2\sum_{k=1}^{p+1}\frac{H_k}{(e^t-1)^{k+1}}\binom{p+1}{k} + (p+1)^2\frac{(t-H_{p+1})e^{(p+1)t}}{(e^t-1)^{p+2}}\\
&= (p+1)^2\sum_{k=1}^{p+1}\frac{H_k}{(e^t-1)^{k+1}}\binom{p+1}{k} + (p+1)^2\frac{(t-H_{p+1})e^{(p+1)t}}{(e^t-1)^{p+2}}\\
&= \frac{(p+1)^2}{p+2}f_{p+1}(t).
\end{aligned}
$$

Therefore Equation (2) holds for all $p \geq 0$ and the elementary proof of Theorem 1 is complete.

References

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