AUTOMATIC EVALUATION OF SUMS OF PRODUCTS OF GENERALIZED FIBONACCI AND LUCAS NUMBERS AND NEW IDENTITIES

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Abstract. It is described how recent sums of products of two Fibonacci(-type) numbers can be computed automatically. More general results (obtained by a computer, with human guidance) are listed and proved by traditional methods.

1. Introduction

The authors of [3, 2, 6] where all concerned with the evaluation of

$$\sum_{k=1}^{n} u_{a+bk}v_{c+dk},$$

where u_k, v_k denote either Fibonacci or Lucas numbers or generalizations of them (recursions of second order, with some parameters).

The aim of the present note is to demonstrate that these evaluations *can be done by a computer.*[1] Consequently, longwinded derivations can be replaced by some instructions for a computer algebra system, and one can get also much more involved sums, not only consisting of 2 factors, as in the cited papers. All that matters is the existence of a *Binet formula* for the numbers involved; giving it to your computer, all that must be done is to sum some geometric series, which is a piece of cake for your friend. We will describe the procedure for *Fibonacci numbers*, but that is no restriction whatsoever; it is merely done to avoid cluttering the notation.

So we have the classical representation

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{with} \quad \alpha, \beta = \frac{1 \pm \sqrt{5}}{2};$$

note that $\alpha\beta = -1$.

What is then left for the human user is (provided it is desired!) to rewrite the outcome of the computers computation which is given in terms of powers of α and β, in terms of Fibonacci numbers. But this is again automatic, using the formulæ

$$\alpha^n = \alpha F_n + F_{n-1}, \qquad \beta^n = -\alpha F_n + F_{n+1}.$$

Typically, the outcome of these replacements is not æsthetically pleasing, and simplifications can be done using the standard recursion for Fibonacci numbers.

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[1]We use Maple.

Using this approach, we computed many more such sums, and we list those that look attractive in the following section.

2. New Fibonacci summations

We get explicit evaluations of

$$X_n^{(d)} = \sum_{k=0}^{n} \prod_{j=0}^{d-1} F_{tk+\lambda j},$$

where t is an even number. Cases must be distinguished between $t \equiv 0 \bmod 4$ and $t \equiv 2 \bmod 4$, and d even or odd. Thus we get four theorems.

Theorem 1. *For $t \equiv 0 \bmod 4$ and even d, we have*

$$X_n^{(d)} = \sum_{j=1}^{d/2} F_{2tnj+((d-1)\lambda+t)j} a_{d,j} + c \cdot n + d,$$

with

$$a_{d,j} = \frac{1}{5^{d/2} F_{tj}} \left\{ \begin{matrix} d \\ \frac{d}{2} - j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d}{2}-j)(\frac{d}{2}-j+1)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Here and in the following we use (for more details see [4, 5]*)*

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\}_\lambda = \frac{\prod\limits_{j=1}^{m} F_{j\lambda}}{\prod\limits_{j=1}^{k} F_{j\lambda} \cdot \prod\limits_{j=1}^{m-k} F_{j\lambda}}.$$

Remark. We do not give an explicit expression for the constants c and d, since they can be recovered from the rest: Assume that we have an identity

$$A(n) = B(n) + c \cdot n + d,$$

then $d = A(0) - B(0)$ and $c = A(1) - A(0) - B(1) + B(0)$. This remark applies also to the following examples.

Theorem 2. *For $t \equiv 0 \bmod 4$ and odd d, we have*

$$X_n^{(d)} = \sum_{j=0}^{(d-1)/2} L_{tn(2j+1)+((d-1)\lambda+t)\frac{2j+1}{2}} a_{d,j} + c \cdot n + d,$$

where

$$a_{d,j} = \frac{1}{5^{(d+1)/2}} F_{\frac{t(2j+1)}{2}} \left\{ \begin{matrix} d \\ \frac{d-1}{2} - j \end{matrix} \right\}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Here and in the following, we use *Lucas numbers* $L_n = \alpha^n + \beta^n$.

Theorem 3. *For $t \equiv 2 \bmod 4$ and even d, we have*

$$X_n^{(d)} = \sum_{j=1}^{d/2} F_{2tnj+((d-1)\lambda+t)j}a_{d,j} + c \cdot n + d,$$

where

$$a_{d,j} = \frac{1}{5^{d/2}F_{4tj}} \begin{Bmatrix} d \\ \frac{d}{2}-j \end{Bmatrix}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

Theorem 4. *For $t \equiv 2 \bmod 4$ and odd d, we have*

$$X_n^{(d)} = \sum_{j=0}^{(d-1)/2} F_{tn(2j+1)+((d-1)\lambda+t)\frac{2j+1}{2}}a_{d,j} + c \cdot n + d,$$

where

$$a_{d,j} = \frac{1}{5^{(d-1)/2}}L_{\frac{t}{2}(2j+1)} \begin{Bmatrix} d \\ \frac{d-1}{2}-j \end{Bmatrix}_\lambda \times \begin{cases} (-1)^{\frac{d-1}{2}-j} & \text{if } \lambda \text{ is even,} \\ (-1)^{\frac{(\frac{d-1}{2}-j)(\frac{d+1}{2}-j)}{2}} & \text{if } \lambda \text{ is odd.} \end{cases}$$

3. Proofs

For fixed t, d, λ, a computer proves the desired identity readily. And, in the first place, that is the way the identities were found. However, for general parameters, we must resort to traditional methods.

We write the product

$$\prod_{j=0}^{d-1} F_{kt+\lambda j}$$

in q-notation:

$$\mathbf{i}^{d(kt-1)+\frac{\lambda d(d-1)}{2}} q^{-\frac{d(kt-1)}{2}-\frac{\lambda d(d-1)}{4}} \frac{(q^{tk};q^\lambda)_d}{(1-q)^d}.$$

Here, we set $q = \beta/\alpha$ and use the notation $(x;p)_n = (1-x)(1-xp)\dots(1-xp^{n-1})$. We will also use

$$\begin{bmatrix} N \\ k \end{bmatrix}_\lambda = \frac{(q^\lambda;q^\lambda)_N}{(q^\lambda;q^\lambda)_k(q^\lambda;q^\lambda)_{N-k}}.$$

By the q-binomial theorem (or Rothe's identity, see [1]) we have

$$(q^{tk};q^\lambda)_d = \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_\lambda (-1)^j q^{\lambda\binom{j}{2}} q^{tkj}$$

and so the product takes the form

$$\prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{\mathbf{i}^{d(kt-1)+\frac{\lambda d(d-1)}{2}} q^{-\frac{d(kt-1)}{2}-\frac{\lambda d(d-1)}{4}}}{(1-q)^d} \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_\lambda (-1)^j q^{\lambda\binom{j}{2}} q^{tkj}.$$

Now we again convert it back into Fibonacci-type form:

$$\prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d} \left\{ {d \atop j} \right\}_\lambda (-1)^j (-1)^{\lambda \frac{j(j-1)}{2}} \alpha^{j(\lambda - d\lambda - 2kt)}.$$

First, we consider the case d is even to prove Theorems 1 and 3.

We reorganize the sum on j:

$$\begin{aligned}
\prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda - d\lambda - 2kt)} \\
&\quad + \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=d/2+1}^{d} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda - d\lambda - 2kt)} + constant \\
&= \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{j(\lambda - d\lambda - 2kt)} \\
&\quad + \frac{\alpha^{dkt+\frac{\lambda d(d-1)}{2}}}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{(d-j)(\lambda(d-j-1)+2)}{2}} \alpha^{(d-j)(\lambda - d\lambda - 2kt)} + constant \\
&= \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda - d\lambda - 2kt)} \\
&\quad + \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \beta^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda - d\lambda - 2kt)} + constant \\
&= \frac{1}{5^{d/2}} \sum_{j=0}^{d/2-1} \left\{ {d \atop j} \right\}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j} L_{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda - d\lambda - 2kt)} + constant \\
&= \frac{1}{5^{d/2}} \sum_{j=1}^{d/2} \left\{ {d \atop \frac{d}{2}-j} \right\}_\lambda (-1)^{\lambda\frac{(\frac{d}{2}-j)(\frac{d}{2}-j-1)}{2}+\frac{d}{2}-j} L_{2ktj+\lambda j(d-1)} + constant.
\end{aligned}$$

Now we have the formula

$$\sum_{k=0}^{n} L_{4ak+b} = \frac{F_{4an+2a+b}}{F_{2a}} + constant, \tag{1}$$

which is easy to derive using the Binet formula.

Therefore we get by summing on k

$$\begin{aligned}
\sum_{k=0}^{n} \prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{1}{5^{d/2}} \sum_{j=1}^{d/2} \left\{ {d \atop \frac{d}{2}-j} \right\}_\lambda (-1)^{\lambda\frac{(\frac{d}{2}-j)(\frac{d}{2}-j-1)}{2}+\frac{d}{2}-j} \frac{F_{2tnj+((d-1)\lambda+t)j}}{F_{tj}} \\
&\quad + constant \cdot n + constant.
\end{aligned}$$

A quick check tells us that this is the desired formula for both parities of λ.

This proof works for Theorem 3 as well, since we only needed that t was even.

In order to prove Theorems 2 and 4, we now assume that d is odd.

$$\begin{aligned}
\prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&\quad + \frac{1}{5^{d/2}} \sum_{j=(d+1)/2}^{d} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&= \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{j(\lambda(j-1)+2)}{2}} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&\quad + \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\frac{(d-j)(\lambda(d-j-1)+2)}{2}} (-1)^{dkt+\frac{\lambda d(d-1)}{2}+(d-j)(\lambda-d\lambda-2kt)} \\
&\quad \times \beta^{-dkt-\frac{\lambda d(d-1)}{2}-(d-j)(\lambda-d\lambda-2kt)} \\
&= \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j} \alpha^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&\quad - \frac{1}{5^{d/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j} \beta^{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&= \frac{1}{5^{(d-1)/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{j(j-1)}{2}+j} F_{dkt+\frac{\lambda d(d-1)}{2}+j(\lambda-d\lambda-2kt)} \\
&= \frac{1}{5^{(d-1)/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ \frac{d-1}{2}-j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j} F_{(2j+1)kt+\lambda(d-1)\frac{2j+1}{2}}.
\end{aligned}$$

We have

$$\sum_{k=0}^{n} F_{4ak+b} = \frac{1}{5}\frac{L_{4an+2a+b}}{F_{2a}} + constant$$

and therefore for $t \equiv 0 \bmod 4$

$$\begin{aligned}
\sum_{k=0}^{n}\prod_{j=0}^{d-1} F_{kt+\lambda j} &= \frac{1}{5^{(d-1)/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ \frac{d-1}{2}-j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j} \\
&\quad \times \sum_{k=0}^{n} F_{(2j+1)kt+\lambda(d-1)\frac{2j+1}{2}} \\
&= \frac{1}{5^{(d+1)/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ \frac{d-1}{2}-j \end{Bmatrix}_\lambda (-1)^{\lambda\frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2}+\frac{d-1}{2}-j}
\end{aligned}$$

$$\times \frac{L_{(2j+1)tn+(2j+1)\frac{t}{2}+\lambda(d-1)\frac{2j+1}{2}}}{F_{(2j+1)\frac{t}{2}}} + constant,$$

which settles Theorem 2.

We have

$$\sum_{k=0}^{n} F_{2(2a+1)k+b} = \frac{F_{2(2a+1)n+2a+1+b}}{L_{2a+1}} + constant$$

and therefore for $t \equiv 2 \bmod 4$

$$\sum_{k=0}^{n} \prod_{j=0}^{d-1} F_{kt+\lambda j} = \frac{1}{5^{(d-1)/2}} \sum_{j=0}^{(d-1)/2} \begin{Bmatrix} d \\ \frac{d-1}{2} - j \end{Bmatrix}_{\lambda} (-1)^{\lambda \frac{(\frac{d-1}{2}-j)(\frac{d-1}{2}-j-1)}{2} + \frac{d-1}{2} - j}$$
$$\times \frac{F_{(2j+1)nt+(\lambda(d-1)+t)\frac{2j+1}{2}}}{L_{(2j+1)\frac{t}{2}}} + constant,$$

which settles Theorem 4.

Thus all formulæ have been proved.

4. Conclusion

We have demonstrated that certain sums over products of Fibonacci numbers (or similar ones) can be evaluated by a computer. Doing so, new identities were discovered, were each individual one can be proved by a computer, but for the general families, we needed human proofs.

The four theorems we gave are in terms of Fibonacci numbers. Of course, all this can be done *mutatis mutandis* for Lucas numbers and similar expressions. We do not encourage other authors to write single papers that contain just 4 similar theorems for Lucas numbers since, once the path has been paved, there isn't much intellectual effort in doing this.

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