# $q$–IDENTITIES OF FU AND LASCOUX PROVED BY THE $q$–RICE FORMULA

HELMUT PRODINGER

Abstract. Two recent $q$–identities of Fu and Lascoux are proved by the $q$–Rice formula.

## 1. Introduction

Fu and Lascoux [5] (answering questions of Corteel and Lovejoy, related to identities in [2]) proved the following two identities:

$$\sum_{i=1}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^{i-1}(x+1)\dots(x+q^{i-1})\frac{q^{mi}}{(1-q^i)^m} = \sum_{i=1}^{n}\big(1-(-x)^i\big)\frac{q^i}{1-q^i}\sum_{i\le i_2\le\dots\le i_m\le n}\frac{q^{i_2}}{1-q^{i_2}}\dots\frac{q^{i_m}}{1-q^{i_m}} \tag{1.1}$$

and

$$\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^{i-1}(x+1)\dots(x+q^{i-1})\frac{q^i}{1-tq^i} = -\frac{(q;q)_n}{(t;q)_{n+1}}\sum_{i=0}^{n}\frac{(t;q)_i}{(q;q)_i}(-xq)^i. \tag{1.2}$$

Here, we use the usual notation $(x;q)_n=(1-x)(1-xq)\dots(1-xq^{n-1})$ and $\begin{bmatrix}n\\k\end{bmatrix}=(q;q)_n/(q;q)_k(q;q)_{n-k}$, see [1].

In this short note, we will provide alternative attractive proofs of these, using the $q$–Rice formula, see [6] for some background and applications. Another proof has been obtained recently by Zeng [7] (added during revision).

## 2. Proof of Identity (1.1)

The $q$–Rice formula [6] allows to write an alternating sum as a contour integral:

$$\sum_{i=1}^{n}\begin{bmatrix}n\\i\end{bmatrix}(-1)^{i-1}q^{\binom{i}{2}}f(q^{-i})=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{(q;q)_n}{(z;q)_{n+1}}f(z)dz,$$

where the curve $\mathcal{C}$ encircles the poles $q^{-1},\dots,q^{-n}$ and no others. For more technical details, see [6]. Under mild conditions, the integral (and thus the sum) can be expressed as the negative sum of the further residues. Thus, the computation of the alternating sum boils down to a residue computation.

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In our application, we must find $f(z)$ such that

$$\begin{aligned} f(q^{-i}) &= (x+1)\dots(x+q^{i-1})\frac{q^{mi}}{(1-q^i)^m}q^{-\binom{i}{2}}\\ &= (1+x)\dots\Big(1+\frac{x}{q^{i-1}}\Big)\frac{1}{(q^{-i}-1)^m}. \end{aligned}$$

Now

$$(1+x)\dots\Big(1+\frac{x}{q^{i-1}}\Big)=\prod_{h\ge1}\frac{1+\frac{xq^h}{q^i}}{1+xq^h}.$$

and thus we take

$$f(z)=\frac{1}{(z-1)^m}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}.$$

The only extra pole is at $z=1$, and so the sum is given by

$$\begin{aligned} \mathsf{SUM} &= -\mathrm{Res}_{z=1}\frac{(q;q)_n}{(z;q)_{n+1}}\frac{1}{(z-1)^m}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}\\ &= -[(z-1)^{-1}]\frac{(q;q)_n}{(z;q)_{n+1}}\frac{1}{(z-1)^m}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}\\ &= [(z-1)^m]\frac{(q;q)_n}{(zq;q)_n}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}\\ &= [w^m]\frac{1}{\big(1-w\frac{q}{1-q}\big)\dots\big(1-w\frac{q^n}{1-q^n}\big)}\prod_{h\ge1}\Big(1+\frac{xwq^h}{1+xq^h}\Big). \end{aligned}$$

It is not hard to see that

$$\prod_{h\ge1}\Big(1+\frac{xwq^h}{1+xq^h}\Big)=1-w\sum_{i\ge1}(-x)^i\frac{q^i}{1-q^i}\prod_{1\le h<i}\Big(1-w\frac{q^h}{1-q^h}\Big).$$

To sketch a proof, let us look at the coefficient of $w^2$:

$$\begin{aligned} \sum_{1\le h_1<h_2}\frac{xq^{h_1}}{1+xq^{h_1}}\frac{xq^{h_2}}{1+xq^{h_2}} &= \sum_{1\le h_1<h_2,\ k_1\ge1}\frac{xq^{h_1}}{1+xq^{h_1}}(-1)^{k_1-1}x^{k_1}q^{h_2k_1}\\ &= \sum_{1\le h_1,\ k_1\ge1}\frac{q^{h_1(k_1+1)}}{1+xq^{h_1}}(-1)^{k_1-1}x^{k_1+1}\frac{q^{k_1}}{1-q^{k_1}}\\ &= \sum_{1\le h_1,\ k_1\ge1,\ k_2\ge0}q^{h_1(k_1+1)}q^{h_1k_2}(-1)^{k_1+k_2-1}x^{k_1+k_2+1}\frac{q^{k_1}}{1-q^{k_1}}\\ &= \sum_{1\le h_1,\ 1\le k_1<k_2}q^{h_1k_2}(-1)^{k_2}x^{k_2}\frac{q^{k_1}}{1-q^{k_1}}\\ &= \sum_{1\le k_1<k_2}(-x)^{k_2}\frac{q^{k_1}}{1-q^{k_1}}\frac{q^{k_2}}{1-q^{k_2}}. \end{aligned}$$

If one does this, say, also for the coefficient of $w^3$, then one quickly discovers the general pattern, and these coefficients are the same as the coefficients of the right side.[1]

Now

$$[w^m]\frac{1}{\big(1-w\frac{q}{1-q}\big)\dots\big(1-w\frac{q^n}{1-q^n}\big)}=\sum_{i=1}^{n}\frac{q^i}{1-q^i}\sum_{i\le i_2\le\dots\le i_m\le n}\frac{q^{i_2}}{1-q^{i_2}}\cdots\frac{q^{i_m}}{1-q^{i_m}}$$

is already known (Dilcher's sum [3, 6]), so we are left to prove that

$$\sum_{i=1}^{n\,(\infty)}(-x)^i\frac{q^i}{1-q^i}\sum_{i\le i_2\le\dots\le i_m\le n}\frac{q^{i_2}}{1-q^{i_2}}\cdots\frac{q^{i_m}}{1-q^{i_m}}$$
$$=[w^m]\frac{1}{\big(1-w\frac{q}{1-q}\big)\dots\big(1-w\frac{q^n}{1-q^n}\big)}w\sum_{i\ge1}(-x)^i\frac{q^i}{1-q^i}\prod_{1\le h<i}\Big(1-w\frac{q^h}{1-q^h}\Big)$$

In terms of generating functions, we should show that

$$\sum_{i=1}^{\infty}(-x)^iwa_i\frac{1}{(1-wa_i)\dots(1-wa_n)}$$
$$=\frac{1}{(1-wa_1)\dots(1-wa_n)}\sum_{i\ge1}(-x)^iwa_i\prod_{1\le h<i}(1-wa_h),$$

where we wrote $a_i=q^i/(1-q^i)$ (but it holds in general). But this in equivalent to

$$\sum_{i=1}^{\infty}(-x)^iwa_i\frac{(1-wa_1)\dots(1-wa_n)}{(1-wa_i)\dots(1-wa_n)}=\sum_{i\ge1}(-x)^iwa_i\prod_{1\le h<i}(1-wa_h),$$

and thus proved.

## 3. Proof of Identity (1.2)

This time we take

$$f(z)=\frac{1}{z-t}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}$$

and write

$$\mathsf{SUM}=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{(q;q)_n}{(z;q)_{n+1}}\frac{1}{z-t}\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}dz.$$

Now we use the $q$–binomial theorem (sometimes called Cauchy's formula):

$$\prod_{h\ge1}\frac{1+xzq^h}{1+xq^h}=\frac{(-xzq;q)_\infty}{(-xq;q)_\infty}=\sum_{m\ge0}\frac{(z;q)_m}{(q;q)_m}(-xq)^m.$$

[1]Robin Chapman (private communication) has provided a simple combinatorial proof by interpreting both sides as $\sum_{\text{partitions }\pi}(-x)^{\text{number of parts of }\pi}(-w)^{\text{number of distinct parts of }\pi}q^{|\pi|}$.

However, for the residues at $z = q^{-i}$, $i = 0, \dots, n$, only the terms for $m \le n$ are relevant. Henceforth we may write

$$\mathsf{SUM} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(q;q)_n}{(z;q)_{n+1}} \frac{1}{z-t} \sum_{m=0}^{n} \frac{(z;q)_m}{(q;q)_m}(-xq)^m dz.$$

For outside residues, there is only one, at $z = t$, and therefore

$$\begin{aligned}\mathsf{SUM} &= -\mathrm{Res}_{z=t} \frac{(q;q)_n}{(z;q)_{n+1}} \frac{1}{z-t} \sum_{m=0}^{n} \frac{(z;q)_m}{(q;q)_m}(-xq)^m \\ &= -\frac{(q;q)_n}{(t;q)_{n+1}} \sum_{m=0}^{n} \frac{(t;q)_m}{(q;q)_m}(-xq)^m.\end{aligned}$$

This is clearly equivalent to the formula of Fu and Lascoux.

## 4. Conclusion

This method works equally well for similar sums, like

$$\mathsf{SUM} = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i-1}(x+1)\dots(x+q^{i-1}) \frac{q^i}{(1-tq^i)^2},$$

with the result

$$\mathsf{SUM} = -\mathrm{Res}_{z=t} \frac{(q;q)_n}{(z;q)_{n+1}} \frac{z}{(z-t)^2} \sum_{m=0}^{n} \frac{(z;q)_m}{(q;q)_m}(-xq)^m.$$

Rice's formula belongs to the realm of divided differences, see [4]. This is what links our method and the one of Fu and Lascoux.

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Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits, 2050 Johannesburg, South Africa

*E-mail address*: `helmut@maths.wits.ac.za`