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# Some combinatorial matrices and their LU-decomposition

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**Abstract:** Three combinatorial matrices were considered and their LU-decompositions were found. This is typically done by (creative) guessing, and the proofs are more or less routine calculations.

**Keywords:** Combinatorial matrix, LU-decomposition, Lehmer's matrix, Fibonacci polynomials

**MSC:** 05A19; 15B36

## 1 Introduction

Combinatorial matrices often have beautiful LU-decompositions, which leads also to easy determinant evaluations. It has become a habit of this author to try this decomposition whenever he sees a new such matrix.

The present paper contains three independent ones collected over the last one or two years.

## 2 A matrix from polynomials with bounded roots

In [11] Kirschenhofer and Thuswaldner evaluated the determinant

$$D_s = \det\Big(\frac{1}{(2l)^2 - t^2(2i-1)^2}\Big)_{1\le i,l\le s}$$

for $t = 1$. Consider the matrix $M$ with entries $1/((2l)^2-t^2(2i-1)^2)$ where $s$ might be a positive integer or infinity. In [11], the transposed matrix was considered, but that is immaterial when it comes to the determinant; we will treat the transposed matrix as well, but the results are slightly uglier.

The aim is to provide a completely elementary evaluation of this determinant which relies on the LU-decomposition $LU = M$, which is obtained by guessing. The additional parameter $t$ helps with guessing and makes the result even more general. We found these results:

$$L_{i,j} = \frac{\prod_{k=1}^{j}\big((2j-1)^2t^2-(2k)^2\big)}{\prod_{k=1}^{j}\big((2i-1)^2t^2-(2k)^2\big)}\frac{(i+j-2)!}{(i-j)!(2j-2)!},$$

$$U_{j,l} = \frac{t^{2j-2}(-1)^j16^{j-1}(2j-2)!}{\prod_{k=1}^{j}\big((2k-1)^2t^2-(2l)^2\big)\prod_{k=1}^{j-1}\big((2j-1)^2t^2-(2k)^2\big)}\frac{(j+l-1)!}{l(l-j)!}.$$

Note that

$$\prod_{k=1}^{j}\big((2i-1)^2t^2-(2k)^2\big) = (-1)^j4^j\frac{\Gamma(j+1-t(i-\frac12))}{\Gamma(1-t(i-\frac12))}\frac{\Gamma(j+1+t(i-\frac12))}{\Gamma(1+t(i-\frac12))}$$

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and

$$\prod_{k=1}^{j}\left((2k-1)^2t^2-(2l)^2\right)=4^j t^{2j}\frac{\Gamma(j+\frac{1}{2}+\frac{l}{t})}{\Gamma(\frac{1}{2}+\frac{l}{t})}\frac{\Gamma(j+\frac{1}{2}-\frac{l}{t})}{\Gamma(\frac{1}{2}-\frac{l}{t})};$$

using these formulæ, $L_{i,j}$ resp. $U_{j,l}$ can be written in terms of Gamma functions.

The proof that indeed $\sum_j L_{i,j}U_{j,l} = M_{i,l}$ is within the reach of computer algebra systems (Zeilberger's algorithm). An old version of Maple (without extra packages) provides this summation.

As a bonus, we also state the inverses matrices:

$$L_{i,j}^{-1}=\frac{\prod\limits_{k=1}^{i-1}((2j-1)^2t^2-(2k)^2)}{\prod\limits_{k=1}^{i-1}((2i-1)^2t^2-(2k)^2)}\frac{(-1)^{i+j}(2i-2)!(2j-1)}{(i+j-1)!(i-j)!}$$

and

$$U_{j,l}^{-1}=\prod_{k=1}^{l-1}\left((2k-1)^2t^2-(2j)^2\right)\prod_{k=1}^{l}\left((2l-1)^2t^2-(2k)^2\right)\frac{(-1)^j 2j^2}{t^{2l-2}(2l-2)!(j+l)!(l-j)!16^{l-1}};$$

the necessary proofs are again automatic.

Consequently the determinant is

$$D_s=\prod_{j=1}^{s}U_{j,j}.$$

For $t=1$, this may be simplified:

$$\begin{aligned}
D_s&=\frac{1}{s!}\prod_{j=1}^{s}\frac{(-1)^j16^{j-1}(2j-2)!(2j-1)!}{\prod_{k=1}^{j}(2k-2j-1)(2k+2j-1)\prod_{k=1}^{j-1}(2j-2k-1)(2j+2k-1)}\\
&=\frac{1}{s!}\prod_{j=1}^{s}\frac{16^{j-1}(2j-1)!^2}{(4j-1)!!(4j-3)!!}=\frac{4^s}{s!}\prod_{j=1}^{s}\frac{32^{j-1}(2j-1)!^4}{(4j-1)!(4j-2)!}\\
&=\frac{4^s\,16^{s(s-1)}}{s!^2}\Big/\prod_{j=1}^{s}\binom{4j}{2j}\binom{4j-2}{2j-1}=\frac{4^s\,16^{s(s-1)}}{s!^2}\Big/\prod_{j=1}^{2s}\binom{2j}{j}\\
&=\frac{16^{s(s-1)}}{s!^2}\Big/\prod_{j=0}^{2s-1}\binom{2j+1}{j};
\end{aligned}$$

the last expression was given in [11]. We used the notation $(2n-1)!!=1\cdot3\cdot5\cdots(2n-1)$.

Now we briefly mention the equivalent results for the transposed matrix:

$$L_{i,j}=\frac{\prod_{k=1}^{j}((2k-1)^2t^2-(2j)^2)}{\prod_{k=1}^{j}((2k-1)^2t^2-(2i)^2)}\frac{(i+j-1)!j}{(i-j)!(2j-1)!i},$$

$$U_{j,l}=\frac{t^{2j-2}(-1)^j16^{j-1}(2j-1)!}{\prod_{k=1}^{j}((2l-1)^2t^2-(2k)^2)\prod_{k=1}^{j-1}((2k-1)^2t^2-(2j)^2)}\frac{(j+l-2)!}{j(l-j)!},$$

$$L_{i,j}^{-1}=\frac{\prod_{k=1}^{i-1}((2k-1)^2t^2-(2j)^2)}{\prod_{k=1}^{i-1}((2k-1)^2t^2-(2i)^2)}\frac{(-1)^{i+j}(2i)!j^2}{(i-j)!(i+j)!i^2},$$

$$U_{j,l}^{-1}=\prod_{k=1}^{l}\left((2k-1)^2t^2-(2l)^2\right)\prod_{k=1}^{l-1}\left((2j-1)^2t^2-(2k)^2\right)\frac{(2j-1)!l!(-1)^j}{t^{2l-2}16^{l-1}(2l-1)!(l+j-1)!(l-j)!(l-1)!}.$$

# 3 Lehmer's tridiagonal matrix

Ekhad and Zeilberger [7] have unearthed Lehmer's [12] tridiagonal $n \times n$ matrix $M = M(n)$ with entries (indexed by $1 \le i, j \le n$)

$$M_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ z^{1/2} q^{(i-1)/2} & \text{if } i = j - 1, \\ z^{1/2} q^{(i-2)/2} & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note the similarity to Schur's determinant

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & xq^{1+m} & & & & \cdots \\ -1 & 1 & xq^{2+m} & & & \cdots \\ & -1 & 1 & xq^{3+m} & & \cdots \\ & & -1 & 1 & xq^{4+m} & \cdots \\ & & & \ddots & \ddots & \ddots \end{vmatrix}$$

that was used to great success in [9]. This success was based on the two recursions

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + xq^{1+m}\,\mathsf{Schur}(xq^2)$$

and, with

$$\mathsf{Schur}(x) = \sum_{n \ge 0} a_n x^n,$$

by

$$a_n = q^n a_n + q^{1+m} q^{2n-2} a_{n-1},$$

leading to

$$a_n = \frac{q^{n^2+mn}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

Schur's (and thus Lehmer's) determinant plays an instrumental part in proving the celebrated Rogers-Ramanujan identities and generalizations.

Lehmer [12] has computed the limit for $n \to \infty$ of the determinant of the matrix $M(n)$. Ekhad and Zeilberger [7] have generalized this result by computing the determinant of the finite matrix $M(n)$. Furthermore, a lively account of how modern computer algebra leads to a solution was given. Most prominently, the celebrated $q$-Zeilberger algorithm [14] and creative guessing were used.

In this section, the determinant in question is obtained by computing the LU-decomposition $LU = M$. This is done with a computer, and the exact form of $L$ and $U$ is obtained by guessing. A proof that this is indeed the LU-decomposition is then a routine calculation. From it, the determinant in question is computed by multiplying the diagonal elements of the matrix $U$. By telescoping, the final result is then quite attractive, as already stated and proved by Ekhad and Zeilberger [7].

We use standard notation [2]: $(x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$, and the Gaussian $q$-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$.

## 3.1 The LU-decomposition of $M$

Let

$$\lambda(j) := \sum_{0 \le k \le j/2} \begin{bmatrix} j-k \\ k \end{bmatrix} (-1)^k q^{k(k-1)} z^k.$$

It follows from the basic recursion of the Gaussian $q$-binomial coefficients [2] that

$$\lambda(j) = \lambda(j-1) - zq^{j-2}\lambda(j-2). \tag{1}$$

Then we have

$$U_{j,j} = \frac{\lambda(j)}{\lambda(j-1)}, \qquad U_{j,j+1} = z^{1/2}q^{(j-1)/2},$$

and all other entries in the $U$-matrix are zero. Further,

$$L_{j,j} = 1, \qquad L_{j+1,j} = z^{1/2}q^{(j-1)/2}\frac{\lambda(j-1)}{\lambda(j)},$$

and all other entries in the $L$-matrix are zero.

The typical element of the product $(LU)_{i,j}$, that is

$$\sum_{1\le k\le n} L_{i,k}U_{k,j}$$

is almost always zero; the exceptions are as follows: If $i = j$, then we get

$$L_{j,j}U_{j,j} + L_{j,j-1}U_{j-1,j} = \frac{\lambda(j) + zq^{j-2}\lambda(j-2)}{\lambda(j-1)} = 1,$$

because of the above recursion (1). If $i = j - 1$, then we get

$$L_{j-1,j-1}U_{j-1,j} + L_{j-1,j-2}U_{j-2,j} = z^{1/2}q^{(j-2)/2},$$

and if $i = j + 1$, then we get

$$L_{j+1,j+1}U_{j+1,j} + L_{j+1,j}U_{j,j} = z^{1/2}q^{(j-1)/2}\frac{\lambda(j-1)}{\lambda(j)}\frac{\lambda(j)}{\lambda(j-1)} = z^{1/2}q^{(j-1)/2}.$$

This proves that indeed $LU = M$. Therefore for the determinant of the Lehmer matrix $M$ we obtain the expression

$$\prod_{j=1}^{n}\frac{\lambda(j)}{\lambda(j-1)} = \frac{\lambda(n)}{\lambda(0)} = \sum_{0\le k\le n/2}\begin{bmatrix} n-k \\ k \end{bmatrix}(-1)^k q^{k(k-1)} z^k.$$

Taking the limit $n \to \infty$, leads to the old result by Lehmer for the determinant of the infinite matrix:

$$\lim_{n\to\infty}\det(M(n)) = \sum_{k\ge 0}\frac{(-1)^k q^{k(k-1)} z^k}{(q;q)_k}.$$

**Remarks.**

1. For $q = 1$, Lehmer's determinant plays a role when enumerating lattice paths (Dyck paths) of bounded height, or planar trees of bounded height, see [6, 8, 10].
2. Recursions as in (1) have been studied in [3, 4, 13] and are linked to so-called Schur polynomials [15].

# 4 Matrices for Fibonacci polynomials

Cigler [5] introduced several matrices that have Fibonacci polynomials as determinants; we will only treat two of them as showcases.

The Fibonacci polynomials are

$$F_n(x) = \sum_h \binom{n-h}{h} x^{n-2h};$$

our answers will come out in terms of the related polynomials

$$f_n = \sum_h \binom{n+h}{2h} X^h$$

where we write $X = x^2$ for simplicity. It is easy to check that

$$f_n = (X + 2)f_{n-1} - f_{n-2},$$

for instance by comparing coefficients.

The first matrix is

$$M = \left(\binom{i-1}{j} X + \binom{i+1}{j+1}\right)_{0\le i,j<n}$$

and we will determine its LU-decomposition $M = LU$.

We obtained

$$L_{i,j} = \frac{\binom{i+1}{j+1}\sum_h \binom{j+h}{2h} X^h + \binom{i}{j}\sum_h \binom{j+h}{2h-1} X^h}{\sum_h \binom{j+1+h}{2h} X^h} = \binom{i}{j} + \binom{i}{j+1}\frac{f_j}{f_{j+1}}$$

and

$$U_{j,j} = \frac{\sum_h \binom{j+1+h}{2h} X^h}{\sum_h \binom{j+h}{2h} X^h} = \frac{f_{j+1}}{f_j},$$

$$U_{j,l} = (-1)^{j+l}\frac{\sum_h \binom{j+h}{2h-1} X^h}{\sum_h \binom{j+h}{2h} X^h} = (-1)^{j+l}\left(\frac{f_{j+1}}{f_j} - 1\right), \quad j < l.$$

For a proof, we do this computation:

$$\begin{aligned}
\sum_j L_{i,j}U_{j,l} &= L_{i,l}U_{l,l} + \sum_{0\le j<l} L_{i,j}U_{j,l} \\
&= \left[\binom{i}{l} + \binom{i}{l+1}\frac{f_l}{f_{l+1}}\right]\frac{f_{l+1}}{f_l} + \sum_{0\le j<l}\left[\binom{i}{j} + \binom{i}{j+1}\frac{f_j}{f_{j+1}}\right](-1)^{j+l}\left(\frac{f_{j+1}}{f_j} - 1\right) \\
&= \binom{i}{l}\frac{f_{l+1}}{f_l} + \binom{i}{l+1} + \sum_{0\le j<l}\binom{i}{j}\frac{f_{j+1}}{f_j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} \\
&\quad - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j+1}\frac{f_j}{f_{j+1}}(-1)^{j+l} \\
&= \binom{i}{l+1} + \sum_{0\le j\le l}\binom{i}{j}\frac{(X+2)f_j - f_{j-1}}{f_j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} \\
&\quad - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j+1}\frac{f_j}{f_{j+1}}(-1)^{j+l} \\
&= \binom{i}{l+1} + (X+2)\sum_{0\le j\le l}\binom{i}{j}(-1)^{j+l} + \sum_{0\le j<l}\binom{i}{j+1}(-1)^{j+l} - \sum_{0\le j<l}\binom{i}{j}(-1)^{j+l} \\
&\quad - \sum_{0\le j\le l}\binom{i}{j}\frac{f_{j-1}}{f_j}(-1)^{j+l} + \binom{i}{l} + \sum_{1\le j\le l}\binom{i}{j}\frac{f_{j-1}}{f_j}(-1)^{j+l} \\
&= X\binom{i-1}{l} + \binom{i}{l+1} + \binom{i}{l} + \sum_{0\le j\le l}\binom{i}{j}(-1)^{j+l} - \sum_{1\le j\le l}\binom{i}{j}(-1)^{j+l} - (-1)^l \\
&= X\binom{i-1}{l} + \binom{i+1}{l+1}.
\end{aligned}$$

The determinant is then $U_{0,0}U_{1,1}\ldots U_{n-1,n-1}$, and by telescoping

$$\sum_h \binom{n+h}{2h} X^h = \sum_h \binom{2n-h}{h} x^{2n-2h} = F_{2n}(x).$$

For completeness, we also factor the transposed matrix as $LU = M^t$:

$$L_{i,j} = (-1)^{i+j} \frac{\sum_h \binom{j+h}{2h-1} X^h}{\sum_h \binom{j+1+h}{2h-1} X^h}, \quad \text{for } j < i,$$
$$L_{i,i} = 1,$$

and

$$U_{j,l} = \frac{\binom{l}{j} \sum_h \binom{j+h}{2h-1} X^h + \binom{l+1}{j+1} \sum_h \binom{j+h}{2h} X^h}{\sum_h \binom{j+h}{2h} X^h}.$$

Now we move to the second matrix:

$$M = \left( \binom{i}{j} X + \binom{i+2}{j+1} \right)_{0 \le i,j < n}.$$

We find

$$L_{i,j} = \frac{\binom{i+1}{j+1} \sum_h \binom{j+1+h}{2h+1} X^h + \binom{i}{j} \sum_h \binom{j+1+h}{2h} X^h}{\sum_h \binom{j+2+h}{2h+1} X^h}$$

and

$$U_{j,j} = \frac{\sum_h \binom{j+2+h}{2h+1} X^h}{\sum_h \binom{j+1+h}{2h+1} X^h},$$
$$U_{j,j+1} = 1, \qquad U_{j,l} = 0 \quad \text{for } l \ge j+2.$$

For a proof, we compute

$$\sum_j L_{i,j} U_{j,l} = \frac{\binom{i+1}{l+1} \sum_h \binom{l+1+h}{2h+1} X^h + \binom{i}{l} \sum_h \binom{l+1+h}{2h} X^h}{\sum_h \binom{l+1+h}{2h+1} X^h} + \frac{\binom{i+1}{l} \sum_h \binom{l+h}{2h+1} X^h + \binom{i}{l-1} \sum_h \binom{l+h}{2h} X^h}{\sum_h \binom{l+1+h}{2h+1} X^h}$$

and

$$\begin{aligned}
\sum_h \binom{l+1+h}{2h+1} X^h \sum_j L_{i,j} U_{j,l} &= \binom{i+2}{l+1} \sum_h \binom{l+1+h}{2h+1} X^h - \binom{i+1}{l} \sum_h \binom{l+1+h}{2h+1} X^h \\
&\quad + \binom{i}{l} \sum_h \binom{l+1+h}{2h} X^h + \binom{i+1}{l} \sum_h \binom{l+h}{2h+1} X^h \\
&\quad + \binom{i+1}{l} \sum_h \binom{l+h}{2h} X^h - \binom{i}{l} \sum_h \binom{l+h}{2h} X^h \\
&= \binom{i+2}{l+1} \sum_h \binom{l+1+h}{2h+1} X^h + \binom{i}{l} \sum_h \binom{l+h}{2h-1} X^h \\
&= \binom{i+2}{l+1} \sum_h \binom{l+1+h}{2h+1} X^h + \binom{i}{l} X \sum_h \binom{l+1+h}{2h+1} X^h
\end{aligned}$$

and therefore

$$\sum_j L_{i,j} U_{j,l} = \binom{i+2}{l+1} + \binom{i}{l} X,$$

as required. The determinant is then

$$\sum_h \binom{n+1+h}{2h+1} X^h = \sum_h \binom{n+1+h}{n-h} X^h = \sum_j \binom{2n+1-j}{j} x^{2n-2j} = x^{-1} F_{2n+1}(x^2).$$

For the transposed matrix $LU = M^t$, we find

$$L_{i,i-1} = \frac{\sum_h \binom{i+h}{2h+1} X^h}{\sum_h \binom{i+1+h}{2h+1} X^h},$$
$$L_{i,i} = 1, \qquad L_{i,j} = 0 \quad \text{for } j < i-1,$$

and

$$U_{j,l} = \frac{\binom{l+1}{j+1} \sum_h \binom{j+1+h}{2h+1} X^h + \binom{l}{j} \sum_h \binom{j+1+h}{2h} X^h}{\sum_h \binom{j+1+h}{2h+1} X^h}.$$

For completeness, we mention another recent paper about matrices and Fibonacci polynomials: [1].

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